

ANALYTIC CONTINUATION OF A BIHOLOMORPHIC MAPPING

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ABSTRACT. We present a new proof and its generalization of Pinchuk's theorem of the analytic continuation of a biholomorphic mapping from a strictly pseudoconvex real-analytic hypersurface to a compact strictly pseudoconvex real-analytic hypersurface.

0. Introduction and Preliminary

The main purpose of this article is to present new proofs and their generalizations of the following three theorems concerning the analytic continuation of a biholomorphic mapping on a strongly pseudoconvex analytic real hypersurface.

First, we concern Vitushkin's theorem of a germ of a biholomorphic mapping(cf. [Vi]) as follows:

Theorem 0.1 (Vitushkin). *Let M, M' be two strongly pseudoconvex analytic real hypersurfaces in \mathbb{C}^{n+1} and p, p' be points respectively on M, M' such that the germ M at the point p and the germ M' at the point p' are biholomorphically equivalent. Then there is a positive real number δ depending only on M, M' and p, p' such that a biholomorphic mapping ϕ on an open connected neighborhood U of the point p is analytically continued to the open ball $B(p; \delta)$ as a biholomorphic mapping whenever*

$$\phi(p) = p' \quad \text{and} \quad \phi(U \cap M) \subset M'.$$

Next, we concern Pinchuk's theorem of the analytic continuation of a biholomorphic mapping on a spherical analytic real hypersurface(cf. [Pi], [CJ]) as follows:

Theorem 0.2 (Pinchuk, Chern-Ji). *Let M be a spherical strongly pseudoconvex analytic real hypersurface in \mathbb{C}^{n+1} , U be an open connected neighborhood of a point $p \in M$, and ϕ be a biholomorphic mapping on U such that $\phi(U \cap M) \subset S^{2n+1}$. Then the mapping ϕ is analytically continued along any path in M as a local biholomorphic mapping.*

Finally, we concern Pinchuk's theorem of the analytic continuation of a biholomorphic mapping on a nonspherical nondegenerate analytic real hypersurface(cf. [Pi], [Vi]) as follows:

Theorem 0.3 (Pinchuk, Ezhov-Kruzhilin-Vitushkin). *Let M, M' be nonspherical strongly pseudoconvex analytic real hypersurfaces in \mathbb{C}^{n+1} such that M' is compact. Suppose that there are an open connected neighborhood U of a point $p \in M$ and a biholomorphic mapping ϕ on U such that*

$$\phi(M \cap U) \subset M'.$$

Then the mapping ϕ is analytically continued along any path on M as a local biholomorphic mapping.

In the following subsections, we provide some preliminary results from the papers [Pa2] and [Pa3]. We have attempted to present the results of this paper in the 18th DaeWoo Workshop at Hanseo University, Korea. The short outline of the main results in this article shall appear in the proceedings of the Daewoo Workshop.

0.1. Existence and uniqueness theorem. We take a coordinate system of $\mathbb{C}^n \times \mathbb{C}$ as follows:

$$z \equiv (z^1, \dots, z^n), \quad w = u + iv \equiv z^{n+1}.$$

A holomorphic mapping ϕ in $\mathbb{C}^n \times \mathbb{C}$ consists of $(n+1)$ holomorphic functions

$$f \equiv (f^1, \dots, f^n), \quad g \equiv f^{n+1}.$$

We keep the notations

$$\langle z, z \rangle \equiv z^1 \bar{z}^1 + \dots + z^n \bar{z}^n$$

and

$$\Delta \equiv \frac{\partial^2}{\partial z^1 \partial \bar{z}^1} + \cdots + \frac{\partial^2}{\partial z^e \partial \bar{z}^e} - \cdots - \frac{\partial^2}{\partial z^n \partial \bar{z}^n}.$$

Then it is known that a nondegenerate analytic hypersurface M is locally biholomorphic to a real hypersurface of the following form(cf. [CM], [Pa2]):

$$v = \langle z, z \rangle + \sum_{s,t \geq 2}^{\infty} F_{st}(z, \bar{z}, u)$$

where

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0.$$

We shall denote by H the isotropy subgroup of a real hyperquadric $v = \langle z, z \rangle$ such that

$$H = \left\{ \begin{pmatrix} \rho & 0 & 0 \\ -\sqrt{|\rho|} U a & \sqrt{|\rho|} U & 0 \\ -r - i\langle a, a \rangle & 2ia^\dagger & 1 \end{pmatrix} : \begin{array}{l} \langle Uz, Uz \rangle = \text{sign}(\rho) \langle z, z \rangle, \quad a \in \mathbb{C}^n, \\ \rho \neq 0, \quad \rho, r \in \mathbb{R} \\ a^\dagger = (\overline{a^1}, \dots, \overline{a^e}, -\overline{a^{e+1}}, \dots, -\overline{a^n}) \end{array} \right\}.$$

Theorem 0.4 (Chern-Moser). *Let M be a nondegenerate analytic real hypersurface in \mathbb{C}^{n+1} defined near the origin by the equation*

$$(0.1) \quad v = \langle z, z \rangle + F(z, \bar{z}, u)$$

where

$$F(z, \bar{z}, u) = o(|z_1|^2 + \cdots + |z_n|^2 + |u|^2).$$

Then, for each element $(U, a, \rho, r) \in H$, there exists a unique biholomorphic mapping $\phi = (f, g)$ near the origin which transforms M to a real hypersurface in Chern-Moser normal form such that

$$\begin{aligned} \left(\frac{\partial f}{\partial z} \Big|_0 \right) &= C, \quad \left(\frac{\partial f}{\partial w} \Big|_0 \right) = -Ca \\ \text{Re} \left(\frac{\partial g}{\partial w} \Big|_0 \right) &= \rho, \quad \text{Re} \left(\frac{\partial^2 g}{\partial w^2} \Big|_0 \right) = 2\rho r \end{aligned}$$

where the constants (U, a, ρ, r) shall be called the initial value of the normalization ϕ .

We present a brief outline of the proof of Theorem 0.4(cf. [CM], [Pa2]). First of all, we show that there is a biholomorphic mapping

$$\phi_1 : \begin{cases} z = z^* + D(z^*, w^*) \\ w = w^* + g(z^*, w^*) \end{cases}$$

which transforms the equation (0.1) to an equation of the form

$$v^* = F_{11}^*(z^*, \bar{z}^*, u^*) + \sum_{s,t \geq 2} F_{st}^*(z^*, \bar{z}^*, u^*).$$

Then we set

$$p(u) \equiv D(0, u)$$

and we verify that the functions

$$D(z, w), \quad g(z, w), \quad F_{st}^*(z, \bar{z}, u)$$

are uniquely determined by the function $F(z, \bar{z}, u)$ and $p(u)$ whenever we require the following normalizing condition

$$\overline{g(0, u)} = -g(0, u).$$

Further, the functions

$$\left(\frac{\partial^{|I|} D}{\partial z^I} \Big|_{z=v=0} \right), \quad \left(\frac{\partial^{|I|} g}{\partial z^I} \Big|_{z=v=0} \right), \quad \left(\frac{\partial^{|I|+|J|} F_{st}^*}{\partial z^I \partial \bar{z}^J} \Big|_{z=\bar{z}=0} \right)$$

depend analytically on u and $p(u)$, rationally on $p'(u)$, and polynomially on the higher order derivatives of $p(u)$.

At this point, we need an operator tr introduced by Chern and Moser as follows:

$$\text{tr} F_{st}^*(z, \bar{z}, u) = \frac{1}{st} \sum_{\alpha, \beta=0}^n h^{\alpha\beta}(u) \left(\frac{\partial^2 F_{st}^*}{\partial z^\alpha \partial \bar{z}^\beta} \right) (z, \bar{z}, u)$$

where

$$F_{11}^*(z, \bar{z}, u) = \sum_{\alpha, \beta=0}^n h_{\alpha\beta}(u) z^\alpha \bar{z}^\beta$$

and $(h^{\alpha\beta}(u))$ is the inverse matrix of $(h_{\alpha\beta}(u))$. Then we show that the equation

$$(\text{tr})^2 F_{23}^*(z, \bar{z}, u) = 0$$

is an ordinary differential equation of the function $p(u)$ as follows:

$$p'' = Q(u, p, \bar{p}, p', \bar{p}').$$

Hence, for a given value $p'(0) \equiv D_w(0, 0) \in \mathbb{C}^n$, there is a unique biholomorphic mapping ϕ_1 which satisfies the normalizaing condition and which transforms the equation (0.1) to an equation of the following form

$$(0.2) \quad v = F_{11}^*(z, \bar{z}, u) + \sum_{s, t \geq 2} F_{st}^*(z, \bar{z}, u)$$

where

$$(\text{tr})^2 F_{23}^*(z, \bar{z}, u) = 0.$$

Note that, for a biholomorphic mapping ϕ , $\phi(0) = 0$, near the origin, there is a unique decomposition

$$\phi = \phi_2 \circ \phi_1$$

where

$$\begin{aligned} \phi_1 &: \begin{cases} z = z^* + D(z^*, w^*) \\ w = w^* + g(z^*, w^*) \end{cases} \\ \phi_2 &: \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\}} q'(w) E(w) z \\ w^* = q(w) \end{cases}. \end{aligned}$$

and where the function $D(z, w), g(z, w), E(w), q(w)$ are complex analytic such that

$$\begin{aligned} D(0, 0) = D_z(0, w) = 0, \quad g(0, 0) = q(0) = 0 \\ \overline{g(0, u)} = -g(0, u), \quad \overline{q(u)} = q(u) \\ \det E(0) \neq 0 \quad \det q'(0) \neq 0. \end{aligned}$$

Let ϕ be the biholomorphic mapping which transforms the equation (0.2) to a defining equation satisfying the following condition

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u)$$

where

$$\Delta^2 G_{23}(z, \bar{z}, u) = 0.$$

Then there is a unique decomposition of a biholomorphic mapping ϕ such that

$$\phi = \phi^* \circ \phi_1$$

where

$$\phi^* : \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\}q'(w)}E(w)z \\ w^* = q(w) \end{cases}$$

and

$$F_{11}^*(z, \bar{z}, u) = \text{sign}\{q'(0)\}\langle E(u)z, E(u)z \rangle.$$

Second, we take a matrix valued function $E_1(u)$ such that

$$F_{11}^*(z, \bar{z}, u) = \langle E_1(u)z, E_1(u)z \rangle.$$

Then there is a biholomorphic mapping

$$\phi_2 : \begin{cases} z^* = E(w)z \\ w^* = w \end{cases}$$

which transforms the equation (0.2) to an equation of the same form

$$v^* = \langle z^*, z^* \rangle + \sum_{s,t \geq 2} G_{st}(z^*, \bar{z}^*, u^*)$$

where

$$\Delta^2 G_{23}(z, \bar{z}, u) = 0.$$

Further, the function $E(u)$ is uniquely determined up to a function $U(u)$ such that

$$E(u) = U(u)E_1(u)$$

where

$$\langle U(u)z, U(u)z \rangle = \langle z, z \rangle.$$

Then we show that the equation

$$\Delta G_{22}(z, \bar{z}, u) = 0$$

is an ordinary differential equation of the function $U(u)$ as follows:

$$U(u)^{-1}U'(u) = R(u).$$

Hence, for a given value $U(0)$, there is a unique biholomorphic mapping ϕ_2 which transforms the equation (0.2) to an equation of the following form

$$(0.3) \quad v = \langle z, z \rangle + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u)$$

where

$$\Delta G_{22}(z, \bar{z}, u) = \Delta^2 G_{23}(z, \bar{z}, u) = 0.$$

Third, we show that there is a biholomorphic mapping

$$\phi_3 : \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\}q'(w)}z \\ w^* = q(w) \end{cases}$$

which transforms the equation (0.3) to an equation of the same form

$$v^* = \langle z^*, z^* \rangle + \sum_{s,t \geq 2} G_{st}^*(z^*, \bar{z}^*, u^*)$$

where

$$\Delta G_{22}^*(z, \bar{z}, u) = \Delta^2 G_{23}^*(z, \bar{z}, u) = 0.$$

Then we show that the equation

$$\Delta^3 G_{33}^*(z, \bar{z}, u) = 0$$

is an ordinary differential equation of the function $q(u)$ as follows:

$$\frac{q'''}{3q'} - \frac{1}{2} \left(\frac{q''}{q'} \right)^2 = \kappa(u).$$

Hence, for given values $q'(0), q''(0)$, there is a unique biholomorphic mapping ϕ_3 which transforms the equation (0.3) to an equation of the following form

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} G_{st}^*(z, \bar{z}, u)$$

where

$$\Delta G_{22}(z, \bar{z}, u) = \Delta^2 G_{23}(z, \bar{z}, u) = \Delta^3 G_{33}(z, \bar{z}, u) = 0.$$

Thus the existence and uniqueness of the biholomorphic mapping ϕ have been reduced to the existence and uniqueness of solutions of the ordinary differential equations, where some constants U, a, ρ, r appear as the initial values of the solutions.

In the paper [Pa2], we have showed that there exist a family of normal forms as follows:

$$v = \frac{1}{4\alpha} \ln \frac{1}{1 - 4\alpha \langle z, z \rangle} + \sum_{s,t \geq 2} F_{st}(z, \bar{z}, u)$$

where $\alpha, \beta \in \mathbb{R}$ and

$$\begin{cases} \Delta F_{22} = \Delta^2 F_{23} = 0 \\ \Delta^3 F_{33} = \beta \Delta^4 (F_{22})^2. \end{cases}$$

In the case of $\alpha = 0$, we assume

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}(z, \bar{z}, u).$$

The value (α, β) is called the type of normal form. Chern-Moser normal form is given in the case of $\alpha = \beta = 0$ and Moser-Vitushkin normal form is defined by taking $\alpha \neq 0$ and $\beta = 0$.

Then each normalization of a real hypersurface M to a normal form of a given type (α, β) is determined by constant initial value parameterized by the local automorphism group H of the following real hypersurface

$$v = \frac{1}{4\alpha} \ln \frac{1}{1 - 4\alpha \langle z, z \rangle},$$

which is locally biholomorphic to a real hyperquadric.

Theorem 0.5. *Let M be a nondegenerate analytic real hypersurface defined by*

$$v = \sum_{k=2}^{\infty} F_k(z, \bar{z}, u).$$

Then there exist unique natural mappings for each $k \geq 2$ such that

$$\begin{aligned} \nu : \{F_l(z, \bar{z}, u) : l \leq k\} \times H \times \mathbb{R}^2 &\longmapsto (f_{k-1}(z, w), g_k(z, w)) \\ \kappa : \{F_l(z, \bar{z}, u) : l \leq k\} \times H \times \mathbb{R}^2 &\longmapsto F_k^*(z, \bar{z}, u) \end{aligned}$$

such that, for a given $\sigma \in H$ and $\alpha, \beta \in \mathbb{R}$, the formal series mapping

$$\phi = \left(\sum_{k=1}^{\infty} f_k(z, w), \sum_{k=2}^{\infty} g_k(z, w) \right)$$

is a biholomorphic normalization of M with initial value $\sigma \in H$ and

$$v = \langle z, z \rangle + \sum_{k=4}^{\infty} F_k^*(z, \bar{z}, u)$$

is the defining equation of the real hypersurface $\phi(M)$ in normal form of type (α, β) .

Then we obtain the following theorem

Theorem 0.6. *Let M be a nondegenerate analytic real hypersurface defined by*

$$v = \sum_{k=2}^{\infty} F_2(z, \bar{z}, u)$$

and $\phi = (\sum_k f_k, \sum_k g_k)$ be a normalization of M such that the real hypersurface $\phi(M)$ is defined in normal form of type (α, β) by the equation

$$v = \langle z, z \rangle + \sum_{k=4}^{\infty} F_k^*(z, \bar{z}, u).$$

Then the functions f_{k-1}, g_k, F_k^ , $k \geq 3$, are given as a finite linear combination of finite multiples of the following factors:*

- (1) *the coefficients of the functions F_l , $l \leq k$,*
- (2) *the constants $C, C^{-1}, \rho, \rho^{-1}, a, r, \alpha, \beta$,*

where (C, a, ρ, r) are the initial value of the normalization ϕ and α, β are the parameters of normal forms.

0.2. Equation of a chain. In the proof of Theorem 0.4, we have a distinguished curve γ on M , which is named a chain by E. Cartan [Ca] and Chern-Moser [CM]. Suppose that there is a nondegenerate analytic real hypersurface M defined near the origin by

$$v = F(z, \bar{z}, u), \quad F|_0 = F_z|_0 = F_{\bar{z}}|_0 = 0.$$

Then there exists an ordinary differential equation

$$(0.4) \quad p'' = Q(u, p, \bar{p}, p', \bar{p}')$$

such that a chain γ , passing through the origin $0 \in M$, is given near the origin by the equation

$$\gamma : \begin{cases} z = p(u) \\ w = u + iF(p(u), \bar{p}(u), u) \end{cases}$$

where $p(u)$ is a solution of the ordinary differential equation (0.4).

The explicit form of the equation (0.4), which depends on the function $F(z, \bar{z}, u)$, is quite complicate(cf. [CM], [Pa2]). Roughly, the function Q in (0.4) is given as follows:

$$(0.5) \quad Q(u, p, \bar{p}, p', \bar{p}') = (A_1 - A_2 \overline{A_1}^{-1} \overline{A_2})^{-1} (B - A_2 \overline{A_1}^{-1} \overline{B})$$

where

- (1) A_1, A_2, B are functions of $u, p, \bar{p}, p', \bar{p}'$,
- (2) A_1, A_2 are $n \times n$ matrices respectively given by

$$\begin{aligned} & [A_1(u, p, \bar{p}, p', \bar{p}')]_{\alpha\beta} \\ &= \left\{ 2iF_{\beta\bar{\alpha}} + 2(1+iF')^{-1} (F'_\beta + iF''F_\beta) F_{\bar{\alpha}} \right\} \times \\ & \quad \left\{ 1 - i(1+iF') F_\gamma p^{\gamma'} + i(1-iF') F_{\bar{\gamma}} \bar{p}^{\bar{\gamma}'} + F'^2 \right\} \\ & \quad - i(1+iF') F_\beta \times \left\{ 2iF_{\gamma\bar{\alpha}} p^{\gamma'} + 2F''F_{\bar{\alpha}} + i(1+iF') F'_{\bar{\alpha}} \right. \\ & \quad \left. + 2(1+iF')^{-1} F_{\bar{\alpha}} (F'_\gamma p^{\gamma'} + iF''F_\gamma p^{\gamma'} + iF''F_{\bar{\gamma}} \bar{p}^{\bar{\gamma}'}) \right\} \end{aligned}$$

and

$$\begin{aligned} & [A_2(u, p, \bar{p}, p', \bar{p}')]_{\alpha\beta} \\ &= 2iF''(1+iF')^{-1} F_{\bar{\alpha}} F_{\bar{\beta}} \times \\ & \quad \left\{ 1 - i(1+iF') F_\gamma p^{\gamma'} + i(1-iF') F_{\bar{\gamma}} \bar{p}^{\bar{\gamma}'} + F'^2 \right\} \\ & \quad + i(1-iF') F_{\bar{\beta}} \times \left\{ 2iF_{\gamma\bar{\alpha}} p^{\gamma'} + 2F''F_{\bar{\alpha}} + i(1+iF') F'_{\bar{\alpha}} \right. \\ & \quad \left. + 2(1+iF')^{-1} F_{\bar{\alpha}} (F'_\gamma p^{\gamma'} + iF''F_\gamma p^{\gamma'} + iF''F_{\bar{\gamma}} \bar{p}^{\bar{\gamma}'}) \right\} \end{aligned}$$

where

$$\begin{aligned} F_\alpha &= \left(\frac{\partial F}{\partial z^\alpha} \right) (p(u), \bar{p}(u), u), \quad F_{\bar{\beta}} = \left(\frac{\partial F}{\partial \bar{z}^\beta} \right) (p(u), \bar{p}(u), u) \\ F' &= \left(\frac{\partial F}{\partial u} \right) (p(u), \bar{p}(u), u), \quad F'' = \frac{1}{2} \left(\frac{\partial^2 F}{\partial u^2} \right) (p(u), \bar{p}(u), u) \\ F'_\alpha &= \left(\frac{\partial^2 F}{\partial z^\alpha \partial u} \right) (p(u), \bar{p}(u), u), \quad F_{\alpha\bar{\beta}} = \left(\frac{\partial^2 F}{\partial z^\alpha \partial \bar{z}^\beta} \right) (p(u), \bar{p}(u), u), \end{aligned}$$

- (3) B is a $n \times 1$ matrix given by at most cubic polynomial with respect to p', \bar{p}' such that B is a finite linear combination of multiples of the derivatives p', \bar{p}' and the following terms:

$$\left(\frac{\partial^{|I|+|J|+m} F}{\partial z^I \partial \bar{z}^J \partial u^m} \right) (p(u), \bar{p}(u), u) \quad \text{for } |I| + |J| + m \leq 5$$

and

$$\left[\det \left\{ (1-iF')^2 F_{\alpha\bar{\beta}} - i(1+iF') F'_\alpha F_{\bar{\beta}} + i(1-iF') F'_{\bar{\beta}} F_\alpha + 2F'' F_\alpha F_{\bar{\beta}} \right\} \right]^{-1}.$$

On the real hyperquadric $v = \langle z, z \rangle$, the chain γ is locally given by

$$\gamma : \begin{cases} z = p(u) \\ w = u + i\langle p(u), p(u) \rangle \end{cases}$$

where $p(u)$ is a solution of the ordinary differential equation(cf. [Pa2]):

$$p'' = \frac{2ip'\langle p', p' \rangle (1 + 3i\langle p, p' \rangle - i\langle p', p \rangle)}{(1 + i\langle p, p' \rangle - i\langle p', p \rangle)(1 + 2i\langle p, p' \rangle - 2i\langle p', p \rangle)}.$$

Further, the chain γ on a real hyperquadric $v = \langle z, z \rangle$ is necessarily given as an intersection of a complex line(cf. [CM], [Pa2]).

Then we may define a chain γ globally. Let M be a nondegenerate analytic real hypersurface and $\gamma : (0, 1) \rightarrow M$ be an open connected curve. Then the curve γ is called a chain if, for each point $p \in \gamma$, there exist an open neighborhood U of the point p and a biholomorphic mapping ϕ which translates the point p to the origin and transforms M to Chern-Moser normal form such that

$$\phi(U \cap \gamma) \subset \{z = v = 0\}.$$

An alternative definition of a chain γ may be given through the intrinsic geometry of nondegenerate real hypersurfaces(cf. [Ca], [CM], [Ta]).

1. NONSINGULAR MATRICES

1.1. A family of nonsingular matrices.

Lemma 1.1. *Let A_m be a matrix as follows:*

$$\begin{pmatrix} 0 & 2 & 0 & \cdots & & 0 \\ m & 3 & 4 & \ddots & & \\ 0 & m-1 & 6 & 6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & 2 & 3m-3 & 2m \\ 0 & & \cdots & 0 & 1 & 3m \end{pmatrix}.$$

Then the eigenvalues of A_m are given by

$$\frac{3m}{2} + \frac{m-2s}{2}\sqrt{17} \quad \text{for } s = 0, \dots, m.$$

Proof. We consider the following system of first order ordinary differential equations:

$$\begin{aligned} y' &= z \\ z' &= 3z + 2y. \end{aligned}$$

Then the general solutions y, z are given by

$$\begin{aligned} y(t) &= c_1 e^{t\lambda_1} + c_2 e^{t\lambda_2}, \\ z(t) &= c_1 \lambda_1 e^{t\lambda_1} + c_2 \lambda_2 e^{t\lambda_2}, \end{aligned} \tag{1.1}$$

where $\lambda_1, \lambda_2, \lambda_1 \neq \lambda_2$, are the two solutions of the quadratic equation:

$$x^2 - 3x - 2 = 0,$$

and c_1, c_2 are arbitrary real numbers.

We take nonzero constants c_1, c_2 so that $y(t), z(t)$ are linear independent. Then we obtain

$$\begin{pmatrix} e^{t\lambda_1} \\ e^{t\lambda_2} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ c_1 \lambda_1 & c_2 \lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}. \tag{1.2}$$

We consider a real vector space V generated by the following elements:

$$y^{m-s}z^s \quad \text{for } s = 0, 1, \dots, m.$$

By the equalities (1.1) and (1.2), the vector space V is generated as well by the following elements:

$$(1.3) \quad \exp t(s\lambda_1 + (m-s)\lambda_2) \quad \text{for } s = 0, 1, \dots, m.$$

We put

$$B_1 = \begin{pmatrix} y^m \\ y^{m-1}z \\ \vdots \\ yz^{m-1} \\ z^m \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} e^{tm\lambda_1} \\ e^{t((m-1)\lambda_1 + \lambda_2)} \\ \vdots \\ e^{t(\lambda_1 + (m-1)\lambda_2)} \\ e^{tm\lambda_2} \end{pmatrix}.$$

Then it is verified that

$$(1.4) \quad \frac{dB_1}{dt} = \begin{pmatrix} 0 & 2 & 0 & \cdots & & 0 \\ m & 3 & 4 & \ddots & & \\ 0 & m-1 & 6 & 6 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & \ddots & 2 & 3m-3 & 2m \\ 0 & & \cdots & 0 & 1 & 3m \end{pmatrix}^T B_1,$$

and

$$\frac{dB_2}{dt} = \begin{pmatrix} m\lambda_1 & 0 & \cdots & & 0 \\ 0 & (m-1)\lambda_1 + \lambda_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda_1 + (m-1)\lambda_2 & 0 \\ 0 & \cdots & & 0 & m\lambda_2 \end{pmatrix} B_2.$$

Hence the derivative $\frac{d}{dt}$ is an endomorphism on V and the vectors in (1.3) are the eigenvectors of the endomorphism $\frac{d}{dt}$. Thus the matrix A_m has eigenvalues as follows:

$$s\lambda_1 + (m-s)\lambda_2 \quad \text{for } s = 0, 1, \dots, m$$

where

$$\lambda_1, \lambda_2 = \frac{3 \pm \sqrt{17}}{2}.$$

This completes the proof. \square

Lemma 1.2. *Let B_m be a matrix as follows:*

$$B_m = \begin{pmatrix} 1 & 2 & 3 & \cdots & m & m+1 \\ m & 7-m & 4 & 0 & \cdots & 0 \\ 0 & m-1 & 10-m & 6 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix}.$$

Then the matrix B_m is nonsingular.

Proof. We easily verify that

$$(1.5) \quad \det B_m = \frac{1}{4} \det C_m$$

where

$$C_m = \begin{pmatrix} 4-m & 2 & 0 & \cdots & 0 \\ m & 7-m & 4 & \ddots & \vdots \\ 0 & m-1 & 10-m & 6 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix}.$$

Note that

$$(1.6) \quad C_m = A_m - (m-4)id_{(m+1) \times (m+1)}.$$

By Lemma 1.1, the eigenvalues of the matrix A_m is given as follows:

$$\frac{3m}{2} + \frac{m-2s}{2}\sqrt{17} \quad \text{for } s = 0, \dots, m.$$

Thus the eigenvalues of the matrix C_m is given by

$$\frac{m+8}{2} + \frac{m-2s}{2}\sqrt{17} \quad \text{for } s = 0, \dots, m.$$

The matrix C_m does not have 0 as its eigenvalue. Therefore the matrix C_m is nonsingular, i.e.,

$$\det C_m \neq 0.$$

By the relation (1.5),

$$\det B_m = \frac{1}{4} \det C_m \neq 0$$

so that the matrix B_m is nonsingular. This completes the proof. \square

1.2. Sufficient condition for Nonsingularity.

Lemma 1.3. *Let $\Delta(m)$, $m \in \mathbb{N}$, denote the function defined as follows:*

$$\Delta(m) = \sum_{k=0}^m \frac{\binom{m}{k}}{k\lambda_1 + (m-k)\lambda_2 - (m-4)} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k}$$

where

$$\lambda_1 = \frac{3 - \sqrt{17}}{2}, \quad \lambda_2 = \frac{3 + \sqrt{17}}{2}.$$

Then

$$\frac{\det E_m(m+1)}{\det E_m(m)} = \Delta(m)^{-1}$$

where

$$E_m(m+1) = \begin{pmatrix} 4-m & 2 & 0 & \cdots & 0 \\ m & 7-m & 4 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & 0 & 1 & 2m+4 \end{pmatrix}$$

and

$$E_m(m) = \begin{pmatrix} 7-m & 4 & 0 & \cdots & 0 \\ m-1 & 10-m & 6 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & 0 & 1 & 2m+4 \end{pmatrix}.$$

Proof. We easily see that

$$\varepsilon = \frac{\det E_m(m+1)}{\det E_m(m)}$$

if and only if the following matrix is singular:

$$\begin{pmatrix} 4-m-\varepsilon & 2 & 0 & \cdots & 0 \\ m & 7-m & 4 & \ddots & \vdots \\ 0 & m-1 & 10-m & 6 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & 0 & 1 & 2m+4 \end{pmatrix}.$$

Then, by the equalities (1.4) and (1.6), there are constants c_s , $s = 0, \dots, m-1$, which are not all zero and satisfy the following equality:

$$(1.7) \quad \sum_{s=0}^{m-1} c_s \frac{d}{dt} \left(y^s z^{m-s} e^{-t(m-4)} \right) = \frac{d}{dt} \left(y^m e^{-t(m-4)} \right) - \varepsilon y^m e^{-t(m-4)}$$

whenever

$$\varepsilon = \frac{\det E_m(m+1)}{\det E_m(m)}.$$

By the expression (1.1), we obtain

$$\begin{aligned} y^m e^{-t(m-4)} &= (c_1 e^{t\lambda_1} + c_2 e^{t\lambda_2})^m e^{-t(m-4)} \\ &= \sum_{k=0}^m \binom{m}{k} c_1^k c_2^{m-k} e^{tk\lambda_1 + t(m-k)\lambda_2} e^{-t(m-4)} \\ &= \frac{d}{dt} \left\{ \sum_{k=0}^m \binom{m}{k} \frac{c_1^k c_2^{m-k} e^{tk\lambda_1 + t(m-k)\lambda_2} e^{-t(m-4)}}{k\lambda_1 + (m-k)\lambda_2 - (m-4)} \right\}. \end{aligned}$$

By using the expression (1.2), we obtain

$$(1.8) \quad y^m e^{-t(m-4)} = \frac{d}{dt} \left\{ \sum_{k=0}^m \frac{\binom{m}{k} \left(\frac{\lambda_2 y - z}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1 y + z}{\lambda_2 - \lambda_1} \right)^{m-k} e^{-t(m-4)}}{k\lambda_1 + (m-k)\lambda_2 - (m-4)} \right\}.$$

Because the derivative $\frac{d}{dt}$ is an isomorphism, the equalities (1.7) and (1.8) yields

$$(1.9) \quad \sum_{s=0}^{m-1} c_s y^s z^{m-s} = y^m - \varepsilon \sum_{k=0}^m \frac{\binom{m}{k} \left(\frac{\lambda_2 y - z}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1 y + z}{\lambda_2 - \lambda_1} \right)^{m-k}}{k\lambda_1 + (m-k)\lambda_2 - (m-4)}.$$

We easily see that the equality (1.9) is satisfied by some constants c_s only if we have the following equality:

$$1 - \varepsilon \sum_{k=0}^m \frac{\binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k}}{k\lambda_1 + (m-k)\lambda_2 - (m-4)} = 0.$$

Thus we have

$$\varepsilon = \Delta(m)^{-1}.$$

This completes the proof. \square

Lemma 1.4. *Let $B_m(2)$, $m \geq 3$, and $B_m(3)$, $m \geq 4$, be matrices as follows:*

$$B_m(2) = \begin{pmatrix} 2 & 3 & 4 & \cdots & m & m+1 \\ m-1 & 7-m & 6 & 0 & \cdots & 0 \\ 0 & m-2 & 10-m & 8 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix}$$

and

$$B_m(3) = \begin{pmatrix} 3 & 4 & 5 & \cdots & m & m+1 \\ m-2 & 10-m & 8 & 0 & \cdots & 0 \\ 0 & m-3 & 13-m & 10 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix}.$$

Then

$$\begin{aligned} \det B_m(2) &\neq 0 \quad \text{if and only if} \quad \Delta(m)^{-1} \neq 4, \\ \det B_m(3) &\neq 0 \quad \text{if and only if} \quad \Delta(m)^{-1} \neq -\frac{4}{3}(m-3). \end{aligned}$$

Proof. We easily verify that

$$\det B_m(2) = \frac{1}{4} \det C_m(2)$$

where

$$C_m(2) = \begin{pmatrix} 9-m & 4 & 0 & \cdots & 0 \\ m-1 & 10-m & 6 & \ddots & \vdots \\ 0 & m-2 & 13-m & 8 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix}.$$

Note that

$$\det C_m(2) = 0$$

if and only if there are numbers c_1, \dots, c_{m-1} satisfying

$$\begin{pmatrix} 9-m & 4 & 0 & \cdots & 0 \\ m-1 & 10-m & 6 & \ddots & \vdots \\ 0 & m-2 & 13-m & 8 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix} \begin{pmatrix} 1 \\ c_1 \\ \vdots \\ c_{m-1} \end{pmatrix} = 0.$$

Then we easily see

$$\begin{pmatrix} -m & 2 & 0 & \cdots & 0 \\ m & 7-m & 4 & \ddots & \vdots \\ 0 & m-1 & 10-m & 6 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix} \begin{pmatrix} \frac{2}{m} \\ 1 \\ c_1 \\ \vdots \\ c_{m-1} \end{pmatrix} = 0$$

so that

$$\det \begin{pmatrix} -m & 2 & 0 & \cdots & 0 \\ m & 7-m & 4 & \ddots & \vdots \\ 0 & m-1 & 10-m & 6 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix} = 0.$$

Hence, by Lemma 1.3, we verify that

$$\det B_m(2) \neq 0$$

if and only if

$$\Delta(m)^{-1} \neq 4.$$

For the case of $B_m(3)$, we easily verify that

$$\det B_m(3) = \frac{1}{4} \det C_m(3)$$

where

$$C_m(3) = \begin{pmatrix} 14-m & 6 & 0 & \cdots & 0 \\ m-2 & 13-m & 8 & \ddots & \vdots \\ 0 & m-3 & 16-m & 10 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix}.$$

Note that

$$\det C_m(3) = 0$$

if and only if there are numbers c_1, \dots, c_{m-2} satisfying

$$\begin{pmatrix} 14-m & 6 & 0 & \cdots & 0 \\ m-2 & 13-m & 8 & \ddots & \vdots \\ 0 & m-3 & 16-m & 10 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix} \begin{pmatrix} 1 \\ c_1 \\ \vdots \\ c_{m-2} \end{pmatrix} = 0.$$

Then we easily see

$$\begin{pmatrix} \frac{m}{3} & 2 & 0 & \cdots & 0 \\ m & 7-m & 4 & \ddots & \vdots \\ 0 & m-1 & 10-m & 6 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix} \begin{pmatrix} -\frac{24}{m(m-1)} \\ \frac{4}{m-1} \\ 1 \\ c_1 \\ \vdots \\ c_{m-2} \end{pmatrix} = 0$$

so that

$$\det \begin{pmatrix} \frac{m}{3} & 2 & 0 & \cdots & 0 \\ m & 7-m & 4 & \ddots & \vdots \\ 0 & m-1 & 10-m & 6 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix} = 0.$$

Hence, by Lemma 1.3, we verify that

$$\det B_m(3) \neq 0$$

if and only if

$$\Delta(m)^{-1} \neq -\frac{4}{3}(m-3).$$

This completes the proof. \square

1.3. Estimates.

Lemma 1.5. *For any two positive integers p, q , the inequality*

$$\left| \sqrt{17} - \frac{p}{q} \right| > \frac{2}{17q^2}$$

is satisfied.

Proof. Note that

$$\left| \sqrt{17} - \frac{p}{q} \right| = \frac{|p^2 - 17q^2|}{\left| \sqrt{17} + \frac{p}{q} \right| q^2} \geq \frac{1}{\left| \sqrt{17} + \frac{p}{q} \right| q^2}.$$

Let c be a positive real number. Then we consider integer pairs (p, q) such that

$$\left| \sqrt{17} - \frac{p}{q} \right| \leq \frac{1}{c},$$

which yields

$$\begin{aligned} \left| \sqrt{17} + \frac{p}{q} \right| &\leq 2\sqrt{17} + \left| \sqrt{17} - \frac{p}{q} \right| \\ &\leq 2\sqrt{17} + \frac{1}{c}. \end{aligned}$$

Thus the inequality

$$\left| \sqrt{17} - \frac{p}{q} \right| > \frac{1}{cq^2}$$

is satisfied for all integer pair (p, q) , $q \geq 1$, by any positive real number c satisfying

$$c > 2\sqrt{17} + \frac{1}{c},$$

i.e.,

$$c > \sqrt{17} + \sqrt{18} = 8.3657 \dots$$

This completes the proof. \square

Lemma 1.6. *Let $F_1(m)$ be a function of $m \in \mathbb{N}$ defined by*

$$F_1(m) \equiv 192m^3 \binom{m}{[0.7m]} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^{[0.7m]} \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m - [0.7m]}.$$

Then

$$F_1(k) \leq F_1(m)$$

whenever

$$m \geq 100 \quad \text{and} \quad k \geq m + 11.$$

Proof. We easily verify that

$$\frac{F_1(m+1)}{F_1(m)} = \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right) \left(1 + \frac{1}{m} \right)^3 \frac{m+1}{[0.7m] + 1}$$

whenever

$$[0.7m] \neq [0.7m + 0.7]$$

and that

$$\frac{F_1(m+1)}{F_1(m)} = \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right) \left(1 + \frac{1}{m} \right)^3 \frac{m+1}{m - [0.7m] + 1}$$

whenever

$$[0.7m] = [0.7m + 0.7].$$

Then we obtain the following estimates:

$$\frac{F_1(m+1)}{F_1(m)} \leq \frac{10}{7} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right) \left(1 + \frac{1}{m} \right)^4$$

whenever

$$[0.7m] \neq [0.7m + 0.7]$$

and that

$$\frac{F_1(m+1)}{F_1(m)} \leq \frac{10}{3} \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right) \left(1 + \frac{1}{m} \right)^4$$

whenever

$$[0.7m] = [0.7m + 0.7].$$

Hence we obtain

$$\begin{aligned} \frac{F_1(m+11)}{F_1(m)} &= \frac{F_1(m+11)}{F_1(m+10)} \times \cdots \times \frac{F_1(m+1)}{F_1(m)} \\ &\leq \frac{10^{10}}{3^3 7^7} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^7 \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^3 \left(1 + \frac{11}{m} \right)^4. \end{aligned}$$

Note that

$$\frac{10}{3} \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right) \leq \frac{10}{7} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)$$

and

$$\begin{aligned} 0.7 \times 1 &= 0.7, & 0.7 \times 2 &= 1.4, \\ 0.7 \times 3 &= 2.1, & 0.7 \times 4 &= 2.8, \\ 0.7 \times 5 &= 3.5, & 0.7 \times 6 &= 4.2, \\ 0.7 \times 7 &= 4.9, & 0.7 \times 8 &= 5.6, \\ 0.7 \times 9 &= 6.3, & 0.7 \times 0 &= 0. \end{aligned}$$

Thus we have the following estimates:

$$\begin{aligned} \frac{F_1(m+12)}{F_1(m)} &\leq A^8 B^3 \left(1 + \frac{12}{m} \right)^4, & \frac{F_1(m+13)}{F_1(m)} &\leq A^8 B^4 \left(1 + \frac{13}{m} \right)^4 \\ \frac{F_1(m+14)}{F_1(m)} &\leq A^8 B^5 \left(1 + \frac{14}{m} \right)^4, & \frac{F_1(m+15)}{F_1(m)} &\leq A^9 B^5 \left(1 + \frac{15}{m} \right)^4 \\ \frac{F_1(m+16)}{F_1(m)} &\leq A^9 B^6 \left(1 + \frac{16}{m} \right)^4, & \frac{F_1(m+17)}{F_1(m)} &\leq A^9 B^7 \left(1 + \frac{17}{m} \right)^4 \\ \frac{F_1(m+18)}{F_1(m)} &\leq A^{10} B^7 \left(1 + \frac{18}{m} \right)^4, & \frac{F_1(m+19)}{F_1(m)} &\leq A^{10} B^8 \left(1 + \frac{19}{m} \right)^4 \\ \frac{F_1(m+20)}{F_1(m)} &\leq A^{10} B^9 \left(1 + \frac{20}{m} \right)^4, & \frac{F_1(m+21)}{F_1(m)} &\leq A^{10} B^{10} \left(1 + \frac{21}{m} \right)^4 \end{aligned}$$

where

$$A = \frac{10}{7} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right), \quad B = \frac{10}{3} \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right).$$

Then the straight forward computation yields

$$\begin{aligned} 65 \geq \max \left\{ \frac{11}{A^{-\frac{7}{4}} B^{-\frac{3}{4}} - 1}, \frac{12}{A^{-\frac{8}{4}} B^{-\frac{3}{4}} - 1}, \frac{13}{A^{-\frac{8}{4}} B^{-\frac{4}{4}} - 1}, \right. \\ \frac{14}{A^{-\frac{8}{4}} B^{-\frac{5}{4}} - 1}, \frac{15}{A^{-\frac{9}{4}} B^{-\frac{5}{4}} - 1}, \frac{16}{A^{-\frac{9}{4}} B^{-\frac{6}{4}} - 1}, \\ \frac{17}{A^{-\frac{9}{4}} B^{-\frac{7}{4}} - 1}, \frac{18}{A^{-\frac{10}{4}} B^{-\frac{7}{4}} - 1}, \frac{19}{A^{-\frac{10}{4}} B^{-\frac{8}{4}} - 1}, \\ \left. \frac{20}{A^{-\frac{10}{4}} B^{-\frac{9}{4}} - 1}, \frac{21}{A^{-\frac{10}{4}} B^{-\frac{10}{4}} - 1} \right\} \end{aligned}$$

so that

$$F_1(k) \leq F_1(m)$$

whenever

$$m \geq 65 \quad \text{and} \quad k \geq m + 11.$$

This completes the proof. \square

Lemma 1.7. *Let $\Delta(m)$ be the function defined in Lemma 1.3. Then*

$$\Delta(m)^{-1} \neq 4, -\frac{4}{3}(m-3) \quad \text{for all } m.$$

Thus the matrices $B_m(2)$, $m \geq 3$, and $B_m(3)$, $m \geq 4$, are nonsingular.

Proof. We define a function δ_m as follows:

$$\Delta(m)^{-1} = (4-m)(1-\delta_m)$$

so that

$$\begin{aligned} (4-m)\Delta(m) &= (1-\delta_m)^{-1} \\ (1.10) \quad &= \sum_{k=0}^m \frac{\binom{m}{k} \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)^k \left(\frac{-\lambda_1}{\lambda_2-\lambda_1}\right)^{m-k}}{1 - \frac{k}{m} \frac{m\lambda_1}{m-4} - (1 - \frac{k}{m}) \frac{m\lambda_2}{m-4}} \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= \frac{3 - \sqrt{17}}{2} = -0.5615\dots \\ \lambda_2 &= \frac{3 + \sqrt{17}}{2} = 3.5615\dots \\ \frac{\lambda_2}{\lambda_2 - \lambda_1} &= \frac{3 + \sqrt{17}}{2\sqrt{17}} = 0.8638\dots \\ \frac{-\lambda_1}{\lambda_2 - \lambda_1} &= \frac{-3 + \sqrt{17}}{2\sqrt{17}} = 0.1361\dots \end{aligned}$$

Note that

$$\sum_{k=0}^m \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)^k \left(\frac{-\lambda_1}{\lambda_2-\lambda_1}\right)^{m-k} = \left(\frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_1}\right)^m = 1.$$

Thus the summation (1.10) is an average of the function

$$\frac{1}{1 - \frac{m\lambda_1}{m-4} \frac{k}{m} - \frac{m\lambda_2}{m-4} (1 - \frac{k}{m})} = \frac{m-4}{m\sqrt{17}} \left(\frac{k}{m} - \frac{1}{2} - \frac{m+8}{2m\sqrt{17}} \right)^{-1}$$

under the binary distribution

$$\binom{m}{k} \left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)^k \left(\frac{-\lambda_1}{\lambda_2-\lambda_1}\right)^{m-k}.$$

In the equation (1.10), the function

$$(1.11) \quad \frac{1}{1 - \frac{m\lambda_1}{m-4} X - \frac{m\lambda_2}{m-4} (1-X)}$$

has a singular point at the value

$$X = \frac{m+8+m\sqrt{17}}{2m\sqrt{17}}.$$

But we easily see that

$$\frac{k}{m} \neq \frac{m+8+m\sqrt{17}}{2m\sqrt{17}}$$

for each $k = 0, \dots, m$.

Then

$$\begin{aligned} & (1 - \delta_m)^{-1} - 1 \\ &= \delta_m (1 - \delta_m)^{-1} \\ &= \sum_{k=0}^m \frac{\frac{k}{m} - \frac{1}{2} - \frac{3}{2\sqrt{17}}}{\frac{k}{m} - \frac{1}{2} - \frac{m+8}{2m\sqrt{17}}} \cdot \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k}. \end{aligned}$$

For $m \geq 13$, we have

$$(1.12) \quad \frac{1}{2} + \frac{m+8}{2m\sqrt{17}} \leq 0.7 \leq \frac{1}{2} + \frac{3}{2\sqrt{17}}$$

so that

$$\begin{aligned} & \sum_{k=0}^m \frac{\left| \frac{k}{m} - \frac{1}{2} - \frac{3}{2\sqrt{17}} \right|}{\left| \frac{k}{m} - \frac{1}{2} - \frac{m+8}{2m\sqrt{17}} \right|} \cdot \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k} \\ &= \sum_{k=0}^{[0.7m]} \frac{\left| \frac{k}{m} - \frac{1}{2} - \frac{3}{2\sqrt{17}} \right|}{\left| \frac{k}{m} - \frac{1}{2} - \frac{m+8}{2m\sqrt{17}} \right|} \cdot \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k} \\ & \quad + \sum_{k=[0.7m]+1}^m \frac{\left| \frac{k}{m} - \frac{1}{2} - \frac{3}{2\sqrt{17}} \right|}{\left| \frac{k}{m} - \frac{1}{2} - \frac{m+8}{2m\sqrt{17}} \right|} \cdot \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k} \end{aligned}$$

where the first summation contains the singular terms as $m \rightarrow \infty$.

By Lemma 1.5, we have the following estimate:

$$\begin{aligned} & \frac{1}{\left| \frac{k}{m} - \frac{1}{2} - \frac{m+8}{2m\sqrt{17}} \right|} \\ &= \frac{34m}{(m+8) \left| \sqrt{17} - \frac{17(2k-m)}{m+8} \right|} \\ (1.13) \quad & < 17^2 m(m+8). \end{aligned}$$

By the inequality (1.12) and the estimate (1.13), we obtain

$$\begin{aligned}
& \sum_{k=0}^m \left| \frac{k}{m} - \frac{1}{2} - \frac{3}{2\sqrt{17}} \right| \cdot \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k} \\
& \leq \frac{17\sqrt{17}(3 + \sqrt{17})}{2} m(m+8) \sum_{k=0}^{[0.7m]} \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k} \\
& \quad + \left(\frac{1}{5} - \frac{m+8}{2m\sqrt{17}} \right)^{-1} \sum_{k=[0.7m]+1}^m \left| \frac{k}{m} - \frac{1}{2} - \frac{3}{2\sqrt{17}} \right| \\
& \quad \times \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k}.
\end{aligned} \tag{1.14}$$

Note that the binary distribution

$$\binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k} \tag{1.15}$$

increases up to the average

$$\frac{k}{m} = \frac{1}{2} + \frac{3}{2\sqrt{17}} \geq 0.7.$$

Thus we obtain, for $m \geq 100$,

$$\begin{aligned}
& \frac{17\sqrt{17}(3 + \sqrt{17})m(m+8)}{2} \sum_{k=0}^{[0.7m]} \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k} \\
& \leq 192m^3 \binom{m}{[0.7m]} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^{[0.7m]} \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-[0.7m]}
\end{aligned}$$

by the following inequality

$$\frac{17\sqrt{17}(3 + \sqrt{17})}{2} [0.7m] m(m+8) \leq 192m^3 \quad \text{for } m \geq 100.$$

By numerical computation, we obtain

$$\begin{aligned}
F_1(100) &= 2114.7 \dots \\
F_1(200) &= 1.5207 \dots \\
F_1(300) &\leq 5.3215 \times 10^{-4}.
\end{aligned}$$

Then, by Lemma 1.6, we obtain the following numerical estimate:

$$F_1(m) \leq 5.33 \times 10^{-4} \quad \text{for } m \geq 400. \tag{1.16}$$

For the second part of the inequality (1.14), we have the following estimate

$$\begin{aligned}
& \sum_{k=0.7m}^m \left| \frac{k}{m} - \frac{1}{2} - \frac{3}{2\sqrt{17}} \right| \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k} \\
& \leq \sqrt{\sum_{k=0}^m \left(\frac{k}{m} - \frac{1}{2} - \frac{3}{2\sqrt{17}} \right)^2 \binom{m}{k} \left(\frac{\lambda_2}{\lambda_2 - \lambda_1} \right)^k \left(\frac{-\lambda_1}{\lambda_2 - \lambda_1} \right)^{m-k}} \\
& = \sqrt{\frac{2}{17m}}.
\end{aligned}$$

We easily verify that

$$F_2(m) \equiv \left(\frac{1}{5} - \frac{m+8}{2m\sqrt{17}} \right)^{-1} \sqrt{\frac{2}{17m}} \searrow 0 \quad \text{as } m \rightarrow \infty.$$

By numerical computation, we obtain

$$(1.17) \quad \begin{aligned} F_2(400) &= 0.2247 \dots \\ F_2(600) &= 0.1815 \dots \\ F_2(800) &= 0.1564 \dots \end{aligned}$$

Note that we have the following estimate

$$|\delta_m| \leq \frac{|F(m)|}{1 - |F(m)|}$$

whenever

$$F(m) \equiv F_1(m) + F_2(m) < 1.$$

Thus we obtain

$$|\delta_m| \leq 0.2$$

whenever

$$|F(m)| \leq \frac{1}{6} = 0.1666 \dots$$

Therefore, by the numerical result in (1.16) and (1.17),

$$|\delta_m| \leq 0.2 \quad \text{for all } m \geq 800.$$

Hence, it suffices to compute the numerical value of the function $\Delta(m)$ up to $m \leq 800$.

Indeed, by numerical computation up to $m = 800$, we can check the tendency of the function δ_m as follows:

$$(1.18) \quad |\delta_m| \leq 0.2$$

for $m \geq 30$ so that

$$\Delta(m)^{-1} \leq -20.$$

Thus we easily see

$$\Delta(m)^{-1} \neq 4, -\frac{4}{3}(m-3) \quad \text{for } m \geq 30.$$

Then we need to check

$$\Delta(m)^{-1} = \frac{\det E_m(m+1)}{\det E_m(m)} \neq 4, -\frac{4}{3}(m-3) \quad \text{for } 1 \leq m \leq 29,$$

or, equivalently,

$$\begin{aligned} \eta(m) &= \frac{\det E_m(m)}{\det E_m(m-1)} = \frac{2m}{4-m-\Delta(m)^{-1}} \\ &\neq -2 \text{ or } 6 \quad \text{for } 1 \leq m \leq 29. \end{aligned}$$

We may compute the value $\eta(m)$ for $1 \leq m \leq 29$ by using the following recurrence relation

$$\det E_m(s+1) = (2m+4-3s) \det E_m(s) - 2s(m-s+1) \det E_m(s-1)$$

for $s = 2, \dots, m$, with the initial values

$$\det E_m(1) = 2m + 4 \quad \text{and} \quad \det E_m(2) = 4(m + 1)^2$$

where

$$E_m(s + 1) = \begin{pmatrix} 2m - 3s + 4 & 2(m - s + 1) & 0 & \cdots & 0 \\ s & 2m - 3s + 7 & 2(m - s + 2) & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 2 & 2m + 1 & 2m \\ 0 & \cdots & 0 & 1 & 2m + 4 \end{pmatrix}.$$

Then we obtain

$$(1.19) \quad \begin{array}{lll} \eta(1) = 2.66 \dots & \eta(11) = -7.14 \dots & \eta(21) = -1.65 \dots \\ \eta(2) = 4.5 & \eta(12) = 39.86 \dots & \eta(22) = -11.21 \dots \\ \eta(3) = 2.75 & \eta(13) = -1.96 \dots & \eta(23) = -31.81 \dots \\ \eta(4) = 0.36 \dots & \eta(14) = -9.84 \dots & \eta(24) = -7.43 \dots \\ \eta(5) = -5.24 \dots & \eta(15) = 19.83 \dots & \eta(25) = -14.96 \dots \\ \eta(6) = 12.05 \dots & \eta(16) = -4.61 \dots & \eta(26) = 118.96 \dots \\ \eta(7) = 0.64 \dots & \eta(17) = -13.56 \dots & \eta(27) = -12.12 \dots \\ \eta(8) = -5.36 \dots & \eta(18) = 6.45 \dots & \eta(28) = -19.25 \dots \\ \eta(9) = 35.53 \dots & \eta(19) = -7.76 \dots & \eta(29) = -3.97 \dots \\ \eta(10) = -0.11 \dots & \eta(20) = -19.12 \dots & \eta(30) = -16.30 \dots \end{array}$$

This completes the proof. \square

2. LOCAL AUTOMORPHISM GROUP OF A REAL HYPERSURFACE

2.1. Polynomial Identities. We shall use the following notations:

$$\begin{aligned} O(k + 1) &= \sum_{s+2t \geq k+1} O(|z|^s |w|^t) \\ O_{\times}(k + 1) &= (O(k), \dots, O(k), O(k + 1)). \end{aligned}$$

Lemma 2.1. *Let M be a nondegenerate analytic real hypersurface defined near the origin by*

$$v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0$$

and ϕ be a biholomorphic mapping near the origin such that the transformed real hypersurface $\phi(M)$ is defined by the equation

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u) + O(k + 1).$$

Suppose that the equation

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u)$$

is in normal form. Then there is a normalization φ of M such that

$$\varphi = \phi + O_{\times}(k + 1).$$

Further, suppose that the normalization φ transforms M to a real hypersurface M' in normal form defined by

$$v = \langle z, z \rangle + F'(z, \bar{z}, u).$$

Then

$$F'(z, \bar{z}, u) = F^*(z, \bar{z}, u) + O(k+1).$$

In the paper [Pa3], we have given the proof of Lemma 2.1.

Lemma 2.2. *Let M be a real hypersurface in normal form defined by the equation*

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u) + O(l+2),$$

where, for all complex number μ ,

$$F_{st}(\mu z, \mu \bar{z}, \mu^2 u) = \mu^l F_{st}(z, \bar{z}, u).$$

Let ϕ be a normalization of M with initial value $(\text{id}_{n \times n}, a, 1, 0) \in H$ such that ϕ transforms M to a real hypersurface in normal form defined by the equation

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u).$$

Then

$$F^*(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u) + O(l+2)$$

if and only if

$$\begin{aligned} & -2i(\langle z, a \rangle - \langle a, z \rangle) \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u) \\ & + \sum_{\min(s-1,t) \geq 2} \sum_{\alpha} \left(\frac{\partial F_{st}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} (u + i\langle z, z \rangle) \\ & \quad + 2i \sum_{t \geq 2} \sum_{\alpha} \left(\frac{\partial F_{2t}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} \langle z, z \rangle \\ & + \sum_{\min(s,t-1) \geq 2} \sum_{\alpha} \left(\frac{\partial F_{st}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} (u - i\langle z, z \rangle) \\ & \quad - 2i \sum_{s \geq 2} \sum_{\alpha} \left(\frac{\partial F_{s2}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} \langle z, z \rangle \\ & + \frac{i}{2} \sum_{\min(s,t) \geq 2} \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \{ \langle z, a \rangle (u + i\langle z, z \rangle) - \langle a, z \rangle (u - i\langle z, z \rangle) \} \\ & \quad + G_{l+1}(z, \bar{z}, u) \\ (2.1) \quad & = 0 \end{aligned}$$

where, for $l = 2k - 1$,

$$\begin{aligned} G_{l+1}(z, \bar{z}, u) &= \frac{g}{2} \{ (k-1) \langle z, z \rangle + iu \} (u + i\langle z, z \rangle)^{k-1} \\ & \quad + \frac{g}{2} \{ (k-1) \langle z, z \rangle - iu \} (u - i\langle z, z \rangle)^{k-1} \end{aligned}$$

and, for $l = 2k$,

$$\begin{aligned} G_{l+1}(z, \bar{z}, u) &= \langle \kappa, z \rangle (u + i\langle z, z \rangle)^k + \langle z, \kappa \rangle (u - i\langle z, z \rangle)^k \\ &\quad + 2ik\langle z, z \rangle \langle z, \kappa \rangle (u + i\langle z, z \rangle)^{k-1} \\ &\quad - 2ik\langle z, z \rangle \langle \kappa, z \rangle (u - i\langle z, z \rangle)^{k-1} \\ &\quad - \langle z, \kappa \rangle (u + i\langle z, z \rangle)^k - \langle \kappa, z \rangle (u - i\langle z, z \rangle)^k \end{aligned}$$

and

$$\begin{aligned} \langle \kappa, z \rangle &= \frac{u^{3-k}}{4k(k-1)(n+1)(n+2)} \left\{ \sum_{\alpha} a^{\alpha} \Delta^2 \left(\frac{\partial F_{33}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \right. \\ &\quad \left. + \sum_{\alpha} \bar{a}^{\alpha} \Delta^2 \left(\frac{\partial F_{24}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \right\} \\ g &= \frac{u^{4-k}}{2k(k-1)(k-2)n(n+1)(n+2)} \left\{ \sum_{\alpha} a^{\alpha} \Delta^3 \left(\frac{\partial F_{43}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \right. \\ &\quad \left. + \sum_{\alpha} \bar{a}^{\alpha} \Delta^3 \left(\frac{\partial F_{34}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \right\}. \end{aligned}$$

Proof. For the initial value $(id_{n \times n}, a, 1, 0) \in H$, we have the following decomposition(cf. [Pa2]):

$$\phi = E \circ \psi$$

where E is a normalization with identity initial value and

$$\psi : \begin{cases} z^* = \frac{z - aw}{1 + 2i\langle z, a \rangle - i\langle a, a \rangle w} \\ w^* = \frac{\bar{\rho}w}{1 + 2i\langle z, a \rangle - i\langle a, a \rangle w} \end{cases}.$$

The mapping ψ transforms M to a real hypersurface M' defined up to $O(l+2)$ by the equation

$$\begin{aligned} v &= \langle z, z \rangle + F_l(z, z, u) \\ &\quad - 2i(\langle z, a \rangle - \langle a, z \rangle)F_l(z, \bar{z}, u) \\ &\quad + \sum_{\alpha} \left(\frac{\partial F_l}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} (u + i\langle z, z \rangle) \\ &\quad + \sum_{\alpha} \left(\frac{\partial F_l}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} (u - i\langle z, z \rangle) \\ &\quad + \frac{i}{2} \left(\frac{\partial F_l}{\partial u} \right) (z, \bar{z}, u) \{ \langle z, a \rangle (u + i\langle z, z \rangle) - \langle a, z \rangle (u - i\langle z, z \rangle) \} \\ &\quad + O(l+2) \end{aligned}$$

where

$$F_l(z, z, u) = \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u).$$

By virtue of Lemma 2.1, we normalize M' up to $O(l+2)$ by a mapping $h = (f, g)$ satisfying

$$\begin{aligned} \left(\frac{\partial f}{\partial z} \Big|_0 \right) &= id_{n \times n}, \quad \left(\frac{\partial f}{\partial w} \Big|_0 \right) = 0, \\ \Re \left(\frac{\partial g}{\partial w} \Big|_0 \right) &= 1, \quad \Re \left(\frac{\partial^2 g}{\partial w^2} \Big|_0 \right) = 0, \end{aligned}$$

so that we obtain

$$\begin{aligned} F^*(z, \bar{z}, u) &= F_l(z, \bar{z}, u) - 2i(\langle z, a \rangle - \langle a, z \rangle) \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u) \\ &+ \sum_{\min(s-1,t) \geq 2} \sum_{\alpha} \left(\frac{\partial F_{st}}{\partial z^\alpha} \right) (z, \bar{z}, u) a^\alpha (u + i\langle z, z \rangle) \\ &+ 2i \sum_{t \geq 2} \sum_{\alpha} \left(\frac{\partial F_{2t}}{\partial z^\alpha} \right) (z, \bar{z}, u) a^\alpha \langle z, z \rangle \\ &+ \sum_{\min(s,t-1) \geq 2} \sum_{\alpha} \left(\frac{\partial F_{st}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, u) \bar{a}^\alpha (u - i\langle z, z \rangle) \\ &- 2i \sum_{s \geq 2} \sum_{\alpha} \left(\frac{\partial F_{s2}}{\partial \bar{z}^\alpha} \right) (z, \bar{z}, u) \bar{a}^\alpha \langle z, z \rangle \\ &+ \frac{i}{2} \sum_{\min(s,t) \geq 2} \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \{ \langle z, a \rangle (u + i\langle z, z \rangle) \\ &\quad - \langle a, z \rangle (u - i\langle z, z \rangle) \} \\ (2.2) \quad &+ G_{l+1}(z, \bar{z}, u) + O(l+2). \end{aligned}$$

where, for $l = 2k - 1$,

$$\begin{aligned} G_{l+1}(z, \bar{z}, u) &= \langle \chi z, z \rangle (u + i\langle z, z \rangle)^{k-1} + \langle z, \chi z \rangle (u - i\langle z, z \rangle)^{k-1} \\ &\quad - \frac{g}{2i} (u + i\langle z, z \rangle)^k + \frac{g}{2i} (u - i\langle z, z \rangle)^k \end{aligned}$$

and, for $l = 2k$,

$$\begin{aligned} G_{l+1}(z, \bar{z}, u) &= \langle \kappa, z \rangle (u + i\langle z, z \rangle)^k + \langle z, \kappa \rangle (u - i\langle z, z \rangle)^k \\ &\quad + 2ik \langle z, z \rangle \langle z, \kappa \rangle (u + i\langle z, z \rangle)^{k-1} \\ &\quad - 2ik \langle z, z \rangle \langle \kappa, z \rangle (u - i\langle z, z \rangle)^{k-1} \\ &\quad - \langle z, \kappa \rangle (u + i\langle z, z \rangle)^k - \langle \kappa, z \rangle (u - i\langle z, z \rangle)^k. \end{aligned}$$

Here the constants χ, g, κ satisfy the conditions

$$\begin{aligned} \langle \chi z, z \rangle + \langle z, \chi z \rangle &= kg \langle z, z \rangle, \\ g &\in \mathbb{R}, \quad \kappa \in \mathbb{C}^n, \end{aligned}$$

and they are uniquely determined by the following conditions:

$$\Delta F_{22}^* = \Delta^2 F_{23}^* = \Delta^3 F_{33}^* = 0,$$

where

$$\begin{aligned} F^*(z, \bar{z}, u) &= \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u) \\ &+ \sum_{\min(s,t) \geq 2} F_{st}^*(z, \bar{z}, u) + O(l+2) \end{aligned}$$

and, for all complex number μ ,

$$F_{st}^*(\mu z, \mu \bar{z}, \mu^2 u) = \mu^{l+1} F_{st}^*(z, \bar{z}, u).$$

Indeed, from the equality (2.2), we obtain

$$\begin{aligned} F_{22}^*(z, \bar{z}, u) &= 2(k-1)i\langle \chi z, z \rangle \langle z, z \rangle u^{k-2} - k(k-1)ig\langle z, z \rangle^2 u^{k-2} \\ F_{23}^*(z, \bar{z}, u) &= -2k(k-1)\langle \kappa, z \rangle \langle z, z \rangle^2 u^{k-2} + 2i\langle a, z \rangle F_{22}(z, \bar{z}, u) \\ &+ 2i \sum_{\alpha} \left(\frac{\partial F_{22}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} \langle z, z \rangle - \frac{i}{2} \left(\frac{\partial F_{22}}{\partial u} \right) (z, \bar{z}, u) \langle a, z \rangle u \\ &+ \sum_{\alpha} \left(\frac{\partial F_{33}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} u + \sum_{\alpha} \left(\frac{\partial F_{24}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} u \\ F_{33}^*(z, \bar{z}, u) &= -\frac{k(k-1)(k-2)}{3} g \langle z, z \rangle^3 u^{k-3} \\ &- 2i\langle z, a \rangle F_{23}(z, \bar{z}, u) + 2i\langle a, z \rangle F_{32}(z, \bar{z}, u) \\ &+ \sum_{\alpha} \left(\frac{\partial F_{43}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} u + i \sum_{\alpha} \left(\frac{\partial F_{32}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} \langle z, z \rangle \\ &+ \sum_{\alpha} \left(\frac{\partial F_{34}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} u - i \sum_{\alpha} \left(\frac{\partial F_{23}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} \langle z, z \rangle \\ &+ \frac{i}{2} \left(\frac{\partial F_{23}}{\partial u} \right) (z, \bar{z}, u) \langle z, a \rangle u - \frac{i}{2} \left(\frac{\partial F_{32}}{\partial u} \right) (z, \bar{z}, u) \langle a, z \rangle u. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \Delta F_{22}^*(z, \bar{z}, u) &= 2(k-1)(n+2)i\langle \chi z, z \rangle u^{k-2} + 2(k-1)i\text{Tr}(\chi)\langle z, z \rangle u^{k-2} \\ &- 2k(k-1)(n+1)ig\langle z, z \rangle^2 u^{k-2} \\ \Delta^2 F_{22}^*(z, \bar{z}, u) &= 4(k-1)(n+1)i\text{Tr}(\chi)u^{k-2} - 2k(k-1)n(n+1)igu^{k-2} \\ \Delta^2 F_{23}^*(z, \bar{z}, u) &= -4k(k-1)(n+1)(n+2)\langle \kappa, z \rangle u^{k-2} \\ &+ \sum_{\alpha} u a^{\alpha} \Delta^2 \left(\frac{\partial F_{33}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) + \sum_{\alpha} u \bar{a}^{\alpha} \Delta^2 \left(\frac{\partial F_{24}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \\ \Delta^3 F_{33}^*(z, \bar{z}, u) &= -2k(k-1)(k-2)n(n+1)(n+2)gu^{k-3} \\ &+ \sum_{\alpha} u a^{\alpha} \Delta^3 \left(\frac{\partial F_{43}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) + \sum_{\alpha} u \bar{a}^{\alpha} \Delta^3 \left(\frac{\partial F_{34}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \end{aligned}$$

Note that the condition $\Delta F_{22}^* = 0$ yields

$$(2.3) \quad 2\langle \chi z, z \rangle = kg\langle z, z \rangle.$$

The condition $\Delta^2 F_{23}^* = \Delta^3 F_{33}^* = 0$, with the equality (2.3), uniquely determines the constants χ, κ, g .

Then we easily see that

$$F^*(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u) + O(l+2)$$

if and only if

$$\sum_{\min(s,t) \geq 2} F_{st}^*(z, \bar{z}, u) = 0.$$

This completes the proof. \square

Note that, for odd integer l ,

$$G_{l+1}(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2, s=t} G_{st}(z, \bar{z}, u)$$

and, for even integer l ,

$$G_{l+1}(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2, s=t \pm 1} G_{st}(z, \bar{z}, u).$$

Lemma 2.3. *Let M be a real hypersurface in normal form defined by the equation*

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + O(l+2)$$

where

$$F_l(z, \bar{z}, u) = \sum_{s \geq 2} F_{ss}(z, \bar{z}, u)$$

and, for all complex number μ ,

$$F_l(\mu z, \mu \bar{z}, \mu^2 u) = \mu^l F_l(z, \bar{z}, u).$$

Let ϕ be a normalization with initial value $(id_{n \times n}, a, 1, 0) \in H$ such that ϕ transforms M to a real hypersurface M' in normal form defined by the equation

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u).$$

Suppose that

$$F^*(z, \bar{z}, u) = F_l(z, \bar{z}, u) + O(l+2).$$

Then there is an identity, for each integer s , $3 \leq s \leq k$, as follows:

$$\sum_{s=2}^k i^{k-s} \langle z, z \rangle^{k-s} \sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \left\{ \langle z, z \rangle \left(\frac{F_{ss}(z, \bar{z}, u)}{u^{k-s}} \right) \right\} = -(2i)^{k-1} \langle \kappa, z \rangle \langle z, z \rangle^k$$

where

$$l = 2k.$$

Proof. By the condition

$$F_l(z, \bar{z}, u) = \sum_{s \geq 2} F_{ss}(z, \bar{z}, u),$$

the identity (2.1) in Lemma 2.2 comes to

$$\begin{aligned}
& 2i(\langle z, a \rangle - \langle a, z \rangle) \sum_{s \geq 2} F_{ss}(z, \bar{z}, u) \\
& - 2i\langle z, z \rangle \sum_{\alpha} \left\{ \left(\frac{\partial F_{22}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} - \left(\frac{\partial F_{22}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} \right\} \\
& - \sum_{s \geq 3} \sum_{\alpha} \left\{ \left(\frac{\partial F_{ss}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} (u + i\langle z, z \rangle) \right. \\
& \quad \left. + \left(\frac{\partial F_{ss}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} (u - i\langle z, z \rangle) \right\} \\
& - \frac{i}{2} \sum_{s \geq 2} \left(\frac{\partial F_{ss}}{\partial u} \right) (z, \bar{z}, u) \{ \langle z, a \rangle (u + i\langle z, z \rangle) - \langle a, z \rangle (u - i\langle z, z \rangle) \} \\
& = \langle \kappa, z \rangle (u + i\langle z, z \rangle)^k + \langle z, \kappa \rangle (u - i\langle z, z \rangle)^k \\
& + 2ik\langle z, z \rangle \langle z, \kappa \rangle (u + i\langle z, z \rangle)^{k-1} - 2ik\langle z, z \rangle \langle \kappa, z \rangle (u - i\langle z, z \rangle)^{k-1} \\
& - \langle z, \kappa \rangle (u + i\langle z, z \rangle)^k - \langle \kappa, z \rangle (u - i\langle z, z \rangle)^k \\
& = \langle \kappa, z \rangle \sum_{t=2}^k \{1 + (-1)^t (2t-1)\} \binom{k}{t} u^{k-t} (i\langle z, z \rangle)^t \\
(2.4) \quad & + \langle z, \kappa \rangle \sum_{t=2}^k \{1 + (-1)^t (2t-1)\} \binom{k}{t} u^{k-t} (-i\langle z, z \rangle)^t
\end{aligned}$$

Then, by Lemma 2.2, the constant κ is given by

$$\langle \kappa, z \rangle = \frac{u^{2-k}}{4k(k-1)(n+1)(n+2)} \sum_{\alpha} u a^{\alpha} \Delta^2 \left(\frac{\partial F_{33}}{\partial z^{\alpha}} \right) (z, \bar{z}, u).$$

By collecting functions of type $(m+2, m+3)$ for $m = 0, \dots, k-2$ in the identity (2.4), we obtain the following identities for each integer s , $3 \leq s \leq k$:

$$\begin{aligned}
& \langle z, z \rangle \left\{ \langle a, z \rangle \left(\frac{\partial F_{s-1, s-1}}{\partial u} \right) (z, \bar{z}, u) + 2i \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{ss}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \right\} \\
& = 4i\langle a, z \rangle F_{ss}(z, \bar{z}, u) + 4i\langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{ss}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \\
& + 2\langle \kappa, z \rangle \{1 + (-1)^s (2s-1)\} \binom{k}{s} u^{k-s} (i\langle z, z \rangle)^s \\
(2.5) \quad & - iu \left\{ \langle a, z \rangle \left(\frac{\partial F_{ss}}{\partial u} \right) (z, \bar{z}, u) + 2i \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{s+1, s+1}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \right\}
\end{aligned}$$

and

$$\begin{aligned}
& 4i\langle a, z \rangle F_{22}(z, \bar{z}, u) + 4i\langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{22}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \\
& - 4k(k-1)\langle \kappa, z \rangle u^{k-2} \langle z, z \rangle^2 \\
& - iu \left\{ \langle a, z \rangle \left(\frac{\partial F_{22}}{\partial u} \right) (z, \bar{z}, u) + 2i \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{33}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \right\} \\
(2.6) \quad & = 0.
\end{aligned}$$

In the equality (2.5), we assume

$$F_{k+1, k+1}(z, \bar{z}, u) = 0.$$

From the equalities (2.5) and (2.6), we obtain the following recurrence relation:

$$\begin{aligned}
A(s) &= iu^{-1} \langle z, z \rangle A(s-1) \\
&+ 4u^{-1} \sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \{ \langle z, z \rangle F_{ss}(z, \bar{z}, u) \} \\
&- 2i \langle \kappa, z \rangle \{ 1 + (-1)^s (2s-1) \} \binom{k}{s} u^{k-s-1} (i \langle z, z \rangle)^s
\end{aligned}$$

for $s = 2, \dots, k$, and

$$A(1) = A(k) = 0.$$

Thus we obtain the following identity:

$$\begin{aligned}
& \sum_{s=2}^k i^{k-s} \langle z, z \rangle^{k-s} \sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \left\{ \langle z, z \rangle \left(\frac{F_{ss}(z, \bar{z}, u)}{u^{k-s}} \right) \right\} \\
&= \frac{i^{k+1}}{2} \langle \kappa, z \rangle \langle z, z \rangle^k \sum_{s=2}^k \{ 1 + (-1)^s (2s-1) \} \binom{k}{s} \\
(2.7) \quad &= -(2i)^{k-1} \langle \kappa, z \rangle \langle z, z \rangle^k.
\end{aligned}$$

This completes the proof. \square

Lemma 2.4. *Suppose that the functions $F_{ss}(z, \bar{z}, u)$, $s = 2, \dots, k$, satisfy the equalities (2.5), where $l = 2k$. Then that the polynomial*

$$F_{ss}(z, \bar{z}, u), \quad s = \max(k-m, 2), \dots, k-1,$$

is divided by $\langle z, z \rangle^{m-k+s}$ whenever

$$a \neq 0$$

and $F_{kk}(z, \bar{z}, u)$ is divided by $\langle z, z \rangle^m$ for $0 \leq m \leq k$.

Proof. The equality (2.5) yields, for $s = 3, \dots, k$,

$$\begin{aligned}
& (k-s+1)\langle z, z \rangle \langle a, z \rangle F_{s-1, s-1}(z, \bar{z}, u) \\
= & -i(k-s-4)\langle a, z \rangle F_{ss}(z, \bar{z}, u) + 2i\langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{ss}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \\
& + 2\langle \kappa, z \rangle \langle z, z \rangle^s \{1 + (-1)^s(2s-1)\} \binom{k}{s} i^s u^{k-s} \\
& + 2u \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{s+1, s+1}}{\partial z^{\alpha}} \right) (z, \bar{z}, u).
\end{aligned}$$

Since $\langle a, z \rangle$ is not a divisor of $\langle z, z \rangle$, this equality yields the desired result. This completes the proof. \square

Lemma 2.5. *Let M be a real hypersurface in normal form defined by the equation*

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + O(l+2)$$

where

$$F_l(z, \bar{z}, u) = \sum_{\min(s, t) \geq 2} F_{st}(z, \bar{z}, u)$$

and, for all complex number μ ,

$$F_l(\mu z, \mu \bar{z}, \mu^2 u) = \mu^l F_l(z, \bar{z}, u).$$

Let ϕ be a normalization with initial value $(id_{n \times n}, a, 1, 0) \in H$ such that ϕ transforms M to a real hypersurface M' in normal form defined by the equation

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u).$$

Suppose that

$$F^*(z, \bar{z}, u) = F_l(z, \bar{z}, u) + O(l+2)$$

and the function $F_l(z, \bar{z}, u)$ contains a nonzero function $F_{st}(z, \bar{z}, u)$ of type (s, t) , $s \neq t$. Then there is an identity, for each integer s , $3 \leq s \leq p$, as follows:

$$\sum_{s=2}^p i^{p-s} \langle z, z \rangle^{p-s} \sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \left\{ \langle z, z \rangle \left(\frac{F_{s, l-2p+s}(z, \bar{z}, u)}{u^{p-s}} \right) \right\} = 0$$

where

$$l-2p = \max \{|t-s| : F_{st}(z, \bar{z}, u) \neq 0\}.$$

Proof. We easily verify that p is an integer satisfying

$$(2.8) \quad 2 \leq p \leq \left\lfloor \frac{l-1}{2} \right\rfloor.$$

By collecting functions of type (s, t) satisfying

$$t-s = l-2p+1$$

in the identity (2.1) in Lemma 2.2, we obtain the following identities for each integer s , $3 \leq s \leq p$:

$$\begin{aligned}
 & \langle z, z \rangle \left\{ \langle a, z \rangle \left(\frac{\partial F_{s-1, l-2p+s-1}}{\partial u} \right) (z, \bar{z}, u) + 2i \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{s, l-2p+s}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \right\} \\
 &= 4i \langle a, z \rangle F_{s, l-2p+s}(z, \bar{z}, u) + 4i \langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{s, l-2p+s}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \\
 (2.9) \quad & -iu \left\{ \langle a, z \rangle \left(\frac{\partial F_{s, l-2p+s}}{\partial u} \right) (z, \bar{z}, u) + 2i \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{s+1, l-2p+s+1}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \right\}
 \end{aligned}$$

and

$$\begin{aligned}
 & 4i \langle a, z \rangle F_{2, l-2p+2}(z, \bar{z}, u) + 4i \langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{2, l-2p+2}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \\
 & -iu \left\{ \langle a, z \rangle \left(\frac{\partial F_{2, l-2p+2}}{\partial u} \right) (z, \bar{z}, u) + 2i \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{3, l-2p+3}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \right\} \\
 (2.10) \quad & = 0.
 \end{aligned}$$

In the equality (2.9), we assume

$$F_{p+1, l-p+1}(z, \bar{z}, u) = 0.$$

From the equalities (2.9) and (2.10), we obtain the following recurrence relation:

$$\begin{aligned}
 A(s) &= iu^{-1} \langle z, z \rangle A(s-1) \\
 &+ 4u^{-1} \sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \{ \langle z, z \rangle F_{s, l-2p+s}(z, \bar{z}, u) \}
 \end{aligned}$$

for $s = 2, \dots, p$, and

$$A(1) = A(p) = 0.$$

Thus we obtain the following identity:

$$(2.11) \quad \sum_{s=2}^p i^{p-s} \langle z, z \rangle^{p-s} \sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \left\{ \langle z, z \rangle \left(\frac{F_{s, l-2p+s}(z, \bar{z}, u)}{u^{p-s}} \right) \right\} = 0.$$

This completes the proof. \square

Lemma 2.6. *Suppose that the functions $F_{st}(z, \bar{z}, u)$ satisfy the equalities (2.9). Then the polynomial*

$$F_{s, l-2p+s}(z, \bar{z}, u), \quad s = \max(p-m, 2), \dots, p-1,$$

is divided by $\langle z, z \rangle^{m-p+s}$ whenever

$$a \neq 0$$

and $F_{p, l-p}(z, \bar{z}, u)$ is divided by $\langle z, z \rangle^m$ for $1 \leq m \leq p$.

Proof. The equality (2.9) yields, for $s = 3, \dots, p$,

$$\begin{aligned} & (p-s+1)\langle z, z \rangle \langle a, z \rangle F_{s-1, l-2p+s-1}(z, \bar{z}, u) \\ = & -(p-s-4)i\langle a, z \rangle F_{s, l-2p+s}(z, \bar{z}, u) + 2i\langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{s, l-2p+s}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) \\ & + 2u \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{s+1, l-2p+s+1}}{\partial z^{\alpha}} \right) (z, \bar{z}, u). \end{aligned}$$

Since $\langle a, z \rangle$ is not a divisor of $\langle z, z \rangle$, this equality yields the desired result. This completes the proof. \square

2.2. Injectivity of a Linear Mapping.

Lemma 2.7. *Let l be a positive integer ≥ 4 and $F_{2, l-2}(z, \bar{z}, 0)$ be a nonzero function of type $(2, l-2)$. Then the following functions*

$$H_{\alpha}(z, \bar{z}, 0) = \frac{\partial}{\partial z^{\alpha}} \{ \langle z, z \rangle F_{2, l-2}(z, \bar{z}, 0) \} \quad \text{for } \alpha = 1, \dots, n,$$

are linearly independent.

Proof. Suppose the functions $H_1(z, \bar{z}, 0), \dots, H_n(z, \bar{z}, 0)$ are linearly dependent over \mathbb{C} . Then there is a nonzero vector $a = (a^{\alpha}) \in \mathbb{C}^n$ such that

$$\sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \{ \langle z, z \rangle F_{2, l-2}(z, \bar{z}, 0) \} = 0.$$

Then we obtain

$$(2.12) \quad \langle a, z \rangle F_{2, l-2}(z, \bar{z}, 0) = \langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{2, l-2}}{\partial z^{\alpha}} \right) (z, \bar{z}, 0).$$

Note that $\langle a, z \rangle$ is not a divisor of $\langle z, z \rangle$ whenever $a \neq 0$. Otherwise there would be a vector $b \in \mathbb{C}^n$ so that

$$\langle z, z \rangle = \langle a, z \rangle \langle z, b \rangle.$$

This is a contradiction to the fact that the hermitian form $\langle z, z \rangle$ is nondegenerate. Hence the polynomial

$$F_{2, l-2}(z, \bar{z}, 0)$$

is divided by $\langle z, z \rangle$ so that there is a polynomial $G_{1, l-3}(z, \bar{z}, 0)$ of type $(1, l-3)$ as follows:

$$F_{2, l-2}(z, \bar{z}, 0) = \langle z, z \rangle G_{1, l-3}(z, \bar{z}, 0).$$

Then the equality (2.12) comes to

$$2\langle a, z \rangle G_{1, l-3}(z, \bar{z}, 0) = \langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{1, l-3}}{\partial z^{\alpha}} \right) (z, \bar{z}, 0).$$

Note that $G_{1, l-3}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle$ as well so that there is a polynomial $G_{0, l-4}(z, \bar{z}, 0)$ as follows:

$$F_{2, l-2}(z, \bar{z}, 0) = \langle z, z \rangle^2 G_{0, l-4}(z, \bar{z}, 0).$$

Then the equality (2.12) comes to

$$(2.13) \quad \langle a, z \rangle G_{0, l-4}(z, \bar{z}, 0) = 0.$$

Note that $\langle a, z \rangle \neq 0$ unless $a = 0$. Thus the equality (2.13) yields

$$F_{2,l-2}(z, \bar{z}, 0) = 0.$$

This is a contradiction to the assumption $F_{2,l-2}(z, \bar{z}, 0) \neq 0$. This completes the proof. \square

Lemma 2.8. *Suppose that*

$$F_l(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where

$$F_l(\mu z, \mu \bar{z}, \mu^2 u) = \mu^l F_l(z, \bar{z}, u)$$

and

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0.$$

Then the linear mapping

$$a \longmapsto H_{l+1}(z, \bar{z}, u; a)$$

is injective, where

$$\begin{aligned} H_{l+1}(z, \bar{z}, u; a) &\equiv -2i(\langle z, a \rangle - \langle a, z \rangle) \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u) \\ &+ \sum_{\min(s-1,t) \geq 2} \sum_{\alpha} \left(\frac{\partial F_{st}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} (u + i\langle z, z \rangle) \\ &\quad + 2i \sum_{t \geq 2} \sum_{\alpha} \left(\frac{\partial F_{2t}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) a^{\alpha} \langle z, z \rangle \\ &+ \sum_{\min(s,t-1) \geq 2} \sum_{\alpha} \left(\frac{\partial F_{st}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} (u - i\langle z, z \rangle) \\ &\quad - 2i \sum_{s \geq 2} \sum_{\alpha} \left(\frac{\partial F_{s2}}{\partial \bar{z}^{\alpha}} \right) (z, \bar{z}, u) \bar{a}^{\alpha} \langle z, z \rangle \\ &+ \frac{i}{2} \sum_{\min(s,t) \geq 2} \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \{ \langle z, a \rangle (u + i\langle z, z \rangle) \\ &\quad - \langle a, z \rangle (u - i\langle z, z \rangle) \} \\ &\quad + G_{l+1}(z, \bar{z}, u) \end{aligned}$$

and $G_{l+1}(z, \bar{z}, u)$ is the function given in Lemma 2.2.

Proof. First, we assume that $l = 2k$ and

$$F_l(z, \bar{z}, u) = \sum_{s=2}^k F_{ss}(z, \bar{z}, u).$$

Suppose that $a \neq 0$ and $F_{kk}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle^m$ for an integer $0 \leq m \leq k$. Then, by Lemma 2.4, there are polynomials

$$G_{k-m, k-m}^s(z, \bar{z}, 0), \quad s = \max(k-m, 2), \dots, k,$$

of type $(k-m, k-m)$ satisfying

$$\frac{F_{ss}(z, \bar{z}, u)}{u^{k-s}} = i^{s-k} \langle z, z \rangle^{m-k+s} G_{k-m, k-m}^s(z, \bar{z}, 0),$$

for

$$\max(k-m, 2) \leq s \leq k.$$

Then from the equality (2.7) we obtain

$$\begin{aligned} & \sum_{s=\max(k-m, 2)}^k (m-k+s+1) \langle a, z \rangle \langle z, z \rangle^m G_{k-m, k-m}^s(z, \bar{z}, 0) \\ = & - \sum_{s=\max(k-m, 2)}^k \langle z, z \rangle^{m+1} \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{k-m, k-m}^s}{\partial z^{\alpha}} \right) (z, \bar{z}, 0) \\ & - \sum_{2 \leq s \leq k-m-1} i^{k-s} \langle z, z \rangle^{k-s} \sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \{ \langle z, z \rangle F_{ss}(z, \bar{z}, 0) \} \\ & - (2i)^{k-1} \langle \kappa, z \rangle \langle z, z \rangle^k. \end{aligned}$$

Hence there are polynomials $A(z, \bar{z}; m)$, $1 \leq m \leq k$, such that

$$(2.14) \quad \sum_{s=\max(k-m, 2)}^k (m-k+s+1) G_{k-m, k-m}^s(z, \bar{z}, 0) = \langle z, z \rangle A(z, \bar{z}; m).$$

The polynomial $A(z, \bar{z}; m)$ for $m = k$ is given by

$$(2.15) \quad \begin{aligned} A(z, \bar{z}; k) &= -(2i)^{k-1} e \langle z, z \rangle^k \\ \langle \kappa, z \rangle &= e \langle a, z \rangle \end{aligned}$$

for some constant e . From the equality (2.5), we obtain for $s = \max(k-m, 2) + 1, \dots, k$,

$$\begin{aligned} & (k-s+1) \langle a, z \rangle \langle z, z \rangle^{m-k+s} G_{k-m, k-m}^{s-1}(z, \bar{z}, 0) \\ & + (4+2m-3k+3s) \langle a, z \rangle \langle z, z \rangle^{m-k+s} G_{k-m, k-m}^s(z, \bar{z}, 0) \\ & + 2(m-k+s+1) \langle a, z \rangle \langle z, z \rangle^{m-k+s} G_{k-m, k-m}^{s+1}(z, \bar{z}, 0) \\ = & -2 \langle z, z \rangle^{m-k+s+1} \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{k-m, k-m}^s}{\partial z^{\alpha}} \right) (z, \bar{z}, 0) \\ & -2 \langle z, z \rangle^{m-k+s+1} \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{k-m, k-m}^{s+1}}{\partial z^{\alpha}} \right) (z, \bar{z}, 0) \\ & + 2 \langle \kappa, z \rangle \{1 + (-1)^s (2s-1)\} \binom{k}{s} (i \langle z, z \rangle)^s. \end{aligned}$$

Thus there are polynomials $B^{s-1}(z, \bar{z}; m)$, $s = \min(k-m, 2) + 1, \dots, k$, such that

$$(2.16) \quad \begin{aligned} & (k-s+1) G_{k-m, k-m}^{s-1}(z, \bar{z}, 0) \\ & + (4-3k+3s+2m) G_{k-m, k-m}^s(z, \bar{z}, 0) \\ & + 2(1-k+s+m) G_{k-m, k-m}^{s+1}(z, \bar{z}, 0) \\ & = \langle z, z \rangle B^{s-1}(z, \bar{z}; m). \end{aligned}$$

The polynomial $B^{s-1}(z, \bar{z}; m)$ for $m = k$ is given by

$$(2.17) \quad \begin{aligned} B^{s-1}(z, \bar{z}; k) &= 2i^s e\{1 + (-1)^s(2s-1)\} \binom{k}{s} \langle z, z \rangle^{k-m-1}, \\ \langle \kappa, z \rangle &= e\langle a, z \rangle. \end{aligned}$$

Hence from the equalities (2.14) and (2.16) we obtain for $k - m \geq 2$

$$(2.18) \quad B_m \begin{pmatrix} G_{k-m, k-m}^{k-m}(z, \bar{z}, 0) \\ G_{k-m, k-m}^{k-m+1}(z, \bar{z}, 0) \\ \vdots \\ G_{k-m, k-m}^k(z, \bar{z}, 0) \end{pmatrix} = \langle z, z \rangle \begin{pmatrix} A(z, \bar{z}; m) \\ B^{k-m}(z, \bar{z}; m) \\ B^{k-m+1}(z, \bar{z}; m) \\ \vdots \\ B^{k-1}(z, \bar{z}; m) \end{pmatrix}$$

where

$$B_m = \begin{pmatrix} 1 & 2 & 3 & \cdots & m & m+1 \\ m & 7-m & 4 & 0 & \cdots & 0 \\ 0 & m-1 & 10-m & 6 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix}.$$

By Lemma 1.2, the equality (2.18) implies that the function $G_{k-m, k-m}^k(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle$ for all $m \leq k-2$. Hence the polynomial $F_{kk}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle^{k-1}$ whenever $a \neq 0$.

Thus $F_{22}(z, \bar{z}, u)$ is divided by $\langle z, z \rangle$. Then the condition $\Delta F_{22} = 0$ implies

$$F_{22}(z, \bar{z}, u) = i^{2-k} u^{k-2} \langle z, z \rangle G_{11}^2(z, \bar{z}, 0) = 0.$$

Then the equalities (2.14) and (2.16) yield

$$(2.19) \quad B_{k-1}(2) \begin{pmatrix} 0 \\ G_{11}^3(z, \bar{z}, 0) \\ \vdots \\ G_{11}^k(z, \bar{z}, 0) \end{pmatrix} = \langle z, z \rangle \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{k-1} \end{pmatrix}$$

where d_1, \dots, d_{k-1} are constants and

$$B_{k-1}(2) = \begin{pmatrix} 2 & 3 & 4 & \cdots & k-1 & k \\ k-2 & 11-k & 6 & 0 & \cdots & 0 \\ 0 & k-3 & 14-k & 8 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2k-1 & 2(k-1) \\ 0 & \cdots & & 0 & 1 & 2k+2 \end{pmatrix}.$$

By Lemma 1.4 and Lemma 1.7, the equality (2.19) implies that the function $G_{11}^k(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle$. Hence the polynomial $F_{kk}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle^k$ whenever $a \neq 0$.

Thus we obtain

$$\begin{aligned} F_{22}(z, \bar{z}, u) &= 0, \\ F_{ss}(z, \bar{z}, u) &= c_s \langle z, z \rangle^s \quad \text{for all } s = 3, \dots, k \end{aligned}$$

where c_s are constant real numbers. By the way, by Lemma 2.2, the constant κ is given by

$$\langle \kappa, z \rangle = \frac{u^{2-k}}{4k(k-1)(n+1)(n+2)} \sum_{\alpha} u a^{\alpha} \Delta^2 \left(\frac{\partial F_{33}}{\partial z^{\alpha}} \right) (z, \bar{z}, u).$$

Because of the condition $\Delta^3 F_{33} = 0$, we obtain

$$F_{33}(z, \bar{z}, u) = 0 \quad \text{and} \quad \kappa = 0$$

whenever $F_{33}(z, \bar{z}, u)$ is divided by $\langle z, z \rangle^3$. Therefore, we have

$$c_3 = \kappa = 0.$$

Thus the equalities (2.15) and (2.17) yield

$$A(z, \bar{z}; k) = B^{s-1}(z, \bar{z}; k) = 0$$

for all $s = 3, \dots, k$. Then the equalities (2.14) and (2.16) yield

$$\begin{pmatrix} 3 & 4 & 5 & \cdots & k & k+1 \\ k-2 & 13-k & 8 & 0 & \cdots & 0 \\ 0 & k-3 & 16-k & 10 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2k+1 & 2k \\ 0 & \cdots & & 0 & 1 & 2k+4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ c_4 \\ \vdots \\ c_{k-1} \\ c_k \end{pmatrix} = 0.$$

Hence we obtain

$$c_4 = \cdots = c_k = 0.$$

This is a contradiction to the assumption

$$F_l(z, \bar{z}, u) \neq 0.$$

Thus we ought to have $a = 0$.

Assume that $F_l(z, \bar{z}, u)$ contains a function $F_{st}(z, \bar{z}, u)$ of type (s, t) , $s \neq t$, so that

$$\begin{aligned} l - 2p &= \max \{ |t - s| : F_{st}(z, \bar{z}, u) \neq 0 \} \\ 2 &\leq p \leq \left\lfloor \frac{l-1}{2} \right\rfloor, \end{aligned}$$

where

$$F_l(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u).$$

Suppose that $p = 2$. Then the equalities (2.9) and (2.10) reduce to

$$4i \langle a, z \rangle F_{2,l-2}(z, \bar{z}, u) + 4i \langle z, z \rangle \sum_{\alpha} a^{\alpha} \left(\frac{\partial F_{2,l-2}}{\partial z^{\alpha}} \right) (z, \bar{z}, u) = 0,$$

where

$$F_{2,l-2}(z, \bar{z}, u) \neq 0.$$

Hence we obtain

$$\sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \{ \langle z, z \rangle F_{2,l-2}(z, \bar{z}, u) \} = 0.$$

By Lemma 2.7, we obtain $a = 0$.

Suppose that

$$3 \leq p \leq \left\lfloor \frac{l-1}{2} \right\rfloor.$$

and

$$F_{p,l-p}(z, \bar{z}, u) = 0.$$

Then by the equalities (2.9) and (2.10), there is a integer m such that

$$\langle z, z \rangle \langle a, z \rangle \left(\frac{\partial F_{m-1,l-m-1}}{\partial u} \right) (z, \bar{z}, u) = 0,$$

where

$$\begin{aligned} 3 &\leq m \leq p, \\ F_{m-1,l-m-1}(z, \bar{z}, u) &\neq 0. \end{aligned}$$

Note that

$$\begin{aligned} \left(\frac{\partial F_{m-1,l-m-1}}{\partial u} \right) (z, \bar{z}, u) &= (p-m+1)u^{-1}F_{m-1,l-m-1}(z, \bar{z}, u) \\ &\neq 0. \end{aligned}$$

Thus we obtain $a = 0$.

Hence we may assume that

$$3 \leq p \leq \left\lfloor \frac{l-1}{2} \right\rfloor.$$

and

$$F_{p,l-p}(z, \bar{z}, u) \neq 0.$$

We claim that $F_{p,l-p}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle^{p-1}$ whenever $a \neq 0$. Suppose that $a \neq 0$ and $F_{p,l-p}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle^m$ for an integer m , $0 \leq m \leq p-2$. Then, by Lemma 2.6, there are polynomials

$$(2.20) \quad G_{p-m,l-p-m}^s(z, \bar{z}, 0), \quad s = \max(p-m, 2), \dots, p,$$

of type $(p-m, l-p-m)$ satisfying

$$\frac{F_{s,l-2p+s}(z, \bar{z}, u)}{u^{p-s}} = i^{s-p} \langle z, z \rangle^{m-p+s} G_{p-m,l-p-m}^s(z, \bar{z}, 0),$$

for

$$\max(p-m, 2) \leq s \leq p.$$

With the polynomials $G_{p-m, l-p-m}^s(z, \bar{z}, 0)$ in (2.20), the equality (2.11) yields

$$\begin{aligned} & \sum_{s=\max(p-m, 2)}^p (m-p+s+1) \langle a, z \rangle \langle z, z \rangle^m G_{p-m, l-p-m}^s(z, \bar{z}, 0) \\ = & - \sum_{s=p-m}^p \langle z, z \rangle^{m+1} \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{p-m, l-p-m}^s}{\partial z^{\alpha}} \right) (z, \bar{z}, 0) \\ & - \sum_{2 \leq s \leq p-m-1} i^{p-s} \langle z, z \rangle^{p-s} \sum_{\alpha} a^{\alpha} \frac{\partial}{\partial z^{\alpha}} \{ \langle z, z \rangle F_{s, l-2p+s}(z, \bar{z}, 0) \}. \end{aligned}$$

Thus there are polynomials $A(z, \bar{z}; m)$, $0 \leq m \leq p-2$, such that

$$(2.21) \quad \sum_{s=p-m}^p (m-p+s+1) G_{p-m, l-p-m}^s(z, \bar{z}, 0) = \langle z, z \rangle A(z, \bar{z}; m).$$

From the equality (2.9), we obtain for $s = p-m+1, \dots, p$,

$$\begin{aligned} & (p-s+1) \langle a, z \rangle \langle z, z \rangle^{m-p+s} G_{p-m, l-p-m}^{s-1}(z, \bar{z}, 0) \\ & + (4+2m-3p+3s) \langle a, z \rangle \langle z, z \rangle^{m-p+s} G_{p-m, l-p-m}^s(z, \bar{z}, 0) \\ & + 2(m-p+s+1) \langle a, z \rangle \langle z, z \rangle^{m-p+s} G_{p-m, l-p-m}^{s+1}(z, \bar{z}, 0) \\ = & -2 \langle z, z \rangle^{m-p+s+1} \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{p-m, l-p-m}^s}{\partial z^{\alpha}} \right) (z, \bar{z}, 0) \\ & -2 \langle z, z \rangle^{m-p+s+1} \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{p-m, l-p-m}^{s+1}}{\partial z^{\alpha}} \right) (z, \bar{z}, 0). \end{aligned}$$

Thus there are polynomials $B^{s-1}(z, \bar{z}; m)$, $s = p-m+1, \dots, p$, such that

$$\begin{aligned} & (p-s+1) G_{p-m, l-p-m}^{s-1}(z, \bar{z}, 0) \\ & + (4-3p+3s+2m) G_{p-m, l-p-m}^s(z, \bar{z}, 0) \\ & + 2(1-p+s+m) G_{p-m, l-p-m}^{s+1}(z, \bar{z}, 0) \\ (2.22) \quad & = \langle z, z \rangle B^{s-1}(z, \bar{z}; m). \end{aligned}$$

Hence, from the equalities (2.21) and (2.22), we obtain

$$(2.23) \quad B_m \begin{pmatrix} G_{p-m, l-p-m}^{p-m} \\ G_{p-m, l-p-m}^{p-m+1} \\ G_{p-m, l-p-m}^{p-m+2} \\ \vdots \\ G_{p-m, l-p-m}^p \end{pmatrix} (z, \bar{z}, 0) = \langle z, z \rangle \begin{pmatrix} A(z, \bar{z}; m) \\ B^{p-m}(z, \bar{z}; m) \\ B^{p-m+1}(z, \bar{z}; m) \\ \vdots \\ B^{p-1}(z, \bar{z}; m) \end{pmatrix}$$

where

$$B_m = \begin{pmatrix} 1 & 2 & 3 & \cdots & m & m+1 \\ m & 7-m & 4 & 0 & \cdots & 0 \\ 0 & m-1 & 10-m & 6 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2m+1 & 2m \\ 0 & \cdots & & 0 & 1 & 2m+4 \end{pmatrix}.$$

By Lemma 1.2, the equality (2.23) implies that the function $G_{p-m, l-p-m}^p(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle$. Hence we prove our claim that $F_{p, l-p}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle^{p-1}$ whenever $a \neq 0$.

Then we claim that $F_{p, l-p}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle^p$ whenever $a \neq 0$. With the polynomials $G_{1, l-2p+1}^s(z, \bar{z}, 0)$ in (2.20), the equality (2.11) yields

$$\langle a, z \rangle \sum_{s=2}^p s G_{1, l-2p+1}^s(z, \bar{z}, 0) = -\langle z, z \rangle \sum_{s=2}^p \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{1, l-2p+1}^s}{\partial z^{\alpha}} \right) (z, \bar{z}, 0).$$

So there is a polynomial $A(z, \bar{z}; p-1)$ of type $(0, l-2p)$ such that

$$(2.24) \quad \sum_{s=2}^p s G_{1, l-2p+1}^s(z, \bar{z}, 0) = \langle z, z \rangle A(z, \bar{z}; p-1).$$

With the polynomials $G_{1, l-2p+1}^s(z, \bar{z}, 0)$ in (2.20), the equality (2.9) yields

$$\begin{aligned} & \langle a, z \rangle \left\{ (p-s+1) G_{1, l-2p+1}^{s-1}(z, \bar{z}, 0) + (2-p+3s) G_{1, l-2p+1}^s(z, \bar{z}, 0) \right. \\ & \quad \left. + 2s G_{1, l-2p+1}^{s+1}(z, \bar{z}, 0) \right\} \\ &= -2 \langle z, z \rangle \left\{ \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{1, l-2p+1}^s}{\partial z^{\alpha}} \right) (z, \bar{z}, 0) + \sum_{\alpha} a^{\alpha} \left(\frac{\partial G_{1, l-2p+1}^{s+1}}{\partial z^{\alpha}} \right) (z, \bar{z}, 0) \right\}. \end{aligned}$$

Then there are polynomials $B^{s-1}(z, \bar{z}; p-1)$ of type $(0, l-2p)$ for $s = 3, \dots, p$ such that

$$(2.25) \quad (p-s+1) G_{1, l-2p+1}^{s-1}(z, \bar{z}, 0) + (2-p+3s) G_{1, l-2p+1}^s(z, \bar{z}, 0) + 2s G_{1, l-2p+1}^{s+1}(z, \bar{z}, 0) = \langle z, z \rangle B^{s-1}(z, \bar{z}; p-1).$$

Hence, from the equalities (2.24) and (2.25), we obtain

$$(2.26) \quad B_{p-1}(2) \begin{pmatrix} G_{1, l-2p+1}^2(z, \bar{z}, 0) \\ G_{1, l-2p+1}^3(z, \bar{z}, 0) \\ \vdots \\ G_{1, l-2p+1}^p(z, \bar{z}, 0) \end{pmatrix} = \langle z, z \rangle \begin{pmatrix} A(z, \bar{z}; p-1) \\ B^2(z, \bar{z}; p-1) \\ B^3(z, \bar{z}; p-1) \\ \vdots \\ B^{p-1}(z, \bar{z}; p-1) \end{pmatrix}$$

where

$$B_{p-1}(2) = \begin{pmatrix} 2 & 3 & 4 & \cdots & p-1 & p \\ p-2 & 11-p & 6 & 0 & \cdots & 0 \\ 0 & p-3 & 14-p & 8 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2p-1 & 2(p-1) \\ 0 & \cdots & & 0 & 1 & 2p+2 \end{pmatrix}.$$

By Lemma 1.4 and Lemma 1.7, the equality (2.26) implies that the polynomial $G_{1, l-2p+1}^p(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle$. Hence we prove our claim that $F_{p, l-p}(z, \bar{z}, 0)$ is divided by $\langle z, z \rangle^p$ whenever $a \neq 0$.

Then with the polynomials $G_{0,l-2p}^s(z, \bar{z}, 0)$ in (2.20), the equality (2.11) yields

$$\langle a, z \rangle \sum_{s=2}^p (s+1) G_{0,l-2p}^s(z, \bar{z}, 0) = 0.$$

Whenever $a \neq 0$, we have

$$(2.27) \quad \sum_{s=2}^p (s+1) G_{1,l-2p+1}^s(z, \bar{z}, 0) = 0.$$

With the polynomials $G_{0,l-2p}^s(z, \bar{z}, 0)$ in (2.20), the equality (2.9) yields

$$\langle a, z \rangle \left\{ (p-s+1) G_{0,l-2p}^{s-1}(z, \bar{z}, 0) + (4-p+3s) G_{0,l-2p}^s(z, \bar{z}, 0) + 2(s+1) G_{0,l-2p}^{s+1}(z, \bar{z}, 0) \right\} = 0.$$

Whenever $a \neq 0$, we have

$$(2.28) \quad (p-s+1) G_{0,l-2p}^{s-1}(z, \bar{z}, 0) + (4-p+3s) G_{0,l-2p}^s(z, \bar{z}, 0) + 2(s+1) G_{0,l-2p}^{s+1}(z, \bar{z}, 0) = 0$$

for $s = 3, \dots, p$. Hence, from the equalities (2.27) and (2.28), we obtain

$$(2.29) \quad B_p(3) \begin{pmatrix} G_{0,l-2p}^2(z, \bar{z}, 0) \\ G_{0,l-2p}^3(z, \bar{z}, 0) \\ \vdots \\ G_{0,l-2p}^p(z, \bar{z}, 0) \end{pmatrix} = 0$$

where

$$B_p(3) = \begin{pmatrix} 3 & 4 & 5 & \cdots & p & p+1 \\ p-2 & 13-p & 8 & 0 & \cdots & 0 \\ 0 & p-3 & 16-p & 10 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 2 & 2p+1 & 2p \\ 0 & \cdots & & 0 & 1 & 2p+4 \end{pmatrix}.$$

By Lemma 1.4 and Lemma 1.7, the equality (2.29) implies

$$G_{0,l-2p}^p(z, \bar{z}, 0) = 0.$$

This is a contradiction to the assumption

$$F_{p,l-p}(z, \bar{z}, u) = \langle z, z \rangle^p G_{0,l-2p}^p(z, \bar{z}, 0) \neq 0.$$

Thus we ought to have $a = 0$ as well for the case of $3 \leq p \leq \left\lfloor \frac{l-1}{2} \right\rfloor$. Therefore we obtain $a = 0$ whenever $F_l(z, \bar{z}, u)$ contains a nonvanishing term $F_{st}(z, \bar{z}, u)$ of type (s, t) , $s \neq t$.

Therefore, we have showed that $a = 0$ whenever

$$F_l(z, \bar{z}, u) \neq 0 \quad \text{and} \quad H_{l+1}(z, \bar{z}, u; a) = 0.$$

This completes the proof. \square

Theorem 2.9. *Let M be a real hypersurface in normal form defined by the equation*

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + O(l+2),$$

where

$$F_l(z, \bar{z}, u) \neq 0$$

and, for all complex numbers μ ,

$$F_l(\mu z, \mu \bar{z}, \mu^2 u) = \mu^l F_l(z, \bar{z}, u).$$

Suppose that there is a normalization ϕ of M with initial value $(id_{n \times n}, a, 1, 0) \in H$ such that ϕ transforms M to a real hypersurface M' defined by the equation

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u)$$

and

$$F^*(z, \bar{z}, u) = F_l(z, \bar{z}, u) + O(l+2).$$

Then the normalization ϕ has identity initial value, i.e., $a = 0$.

Proof. The conclusion follows from Lemma 2.2 and 2.8. □

2.3. Beloshapka-Loboda Theorem.

Lemma 2.10. *Let M be a real hypersurface in normal form and ϕ_{σ_1} be a normalization of M with initial value $\sigma_1 \in H$. Suppose that M is transformed to M' by the normalization ϕ_{σ_1} and ϕ_{σ_2} is a normalization of M' with initial value $\sigma_2 \in H$. Then*

$$\phi_{\sigma_1} \circ \phi_{\sigma_2} = \phi_{\sigma_1 \sigma_2}$$

where $\phi_{\sigma_1 \sigma_2}$ is a normalization of M with initial value $\sigma_1 \sigma_2 \in H$.

In the paper [Pa3], we have given the proof of Lemma 2.10.

Lemma 2.11. *Let M be a real hypersurface in normal form defined by the equation*

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + O(l+2),$$

where

$$F_l(z, \bar{z}, u) \neq 0$$

and, for all complex numbers μ ,

$$F_l(\mu z, \mu \bar{z}, \mu^2 u) = \mu^l F_l(z, \bar{z}, u).$$

Suppose that there is a normalization ϕ of M such that $\phi(M)$ is defined by the equation

$$v = \langle z, z \rangle + \rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) + O(l+2)$$

where

$$\sigma(\phi) = (C, a, \rho, r) \in H.$$

Then the normalization ϕ have the initial value $(C, 0, \rho, r) \in H$, i.e., $a = 0$.

Proof. Note that there is a decomposition of ϕ as follows(cf. [Pa2]):

$$\phi = \phi_{\sigma_1} \circ \phi_{\sigma_2}$$

where $\phi_{\sigma_1}, \phi_{\sigma_2}$ are normalizations with the initial values σ_1, σ_2 respectively:

$$\begin{aligned}\sigma_1 &= (C, 0, \rho, r) \in H, \\ \sigma_2 &= (id_{n \times n}, a, 1, 0) \in H.\end{aligned}$$

Then, by Lemma 2.10, we obtain

$$\begin{aligned}\phi_{\sigma_2} &= \phi_{\sigma_1^{-1}} \circ \phi_{\sigma_1} \circ \phi_{\sigma_2} \\ &= \phi_{\sigma_1^{-1}} \circ \phi\end{aligned}$$

where $\phi_{\sigma_1^{-1}}$ is a normalization with initial value $\sigma_1^{-1} \in H$. Further, suppose that $\phi_{\sigma_2}(M)$ is defined by the equation

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u).$$

Then we obtain

$$F^*(z, \bar{z}, u) = F_l(z, \bar{z}, u) + O(l+2).$$

Thus, by Lemma 2.8, we obtain

$$a = 0.$$

This completes the proof. \square

Theorem 2.12 (Beloshapka, Loboda, Vitushkin). *Let M be an analytic real hypersurface in normal form, which is not a real hyperquadric, and $H(M)$ be the isotropy subgroup of M at the origin. Then there are functions*

$$\rho(U), \quad a(U), \quad r(U)$$

on the set

$$\{U : (U, a, \rho, r) \in H(M) \subset H\}$$

such that, for all $(U, a, \rho, r) \in H(M)$,

$$a = a(U), \quad \rho = \rho(U), \quad r = r(U).$$

Proof. Suppose that M is defined in normal form by the equation

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + F_{l+1}(z, \bar{z}, u) + F_{l+2}(z, \bar{z}, u) + O(l+3),$$

where

$$F_l(z, \bar{z}, u) \neq 0,$$

and the integers $l, l+1, l+2$ represent the weight of the functions

$$F_l(z, \bar{z}, u), \quad F_{l+1}(z, \bar{z}, u), \quad F_{l+2}(z, \bar{z}, u).$$

Let ϕ_σ be a normalization of M with initial value $\sigma \in H(M)$. Suppose that the real hypersurface $\phi_\sigma(M)$ is defined near the origin up to weight l by the equation

$$\begin{aligned}v &= \langle z, z \rangle + \rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) + O(l+1) \\ &= \langle z, z \rangle + F_l(z, \bar{z}, u) + O(l+1)\end{aligned}$$

where

$$\sigma = (C, a, \rho, r) \in H(M) \subset H.$$

Then we have

$$(2.30) \quad |\rho|^{\frac{l-2}{2}} F_l(z, \bar{z}, u) = \lambda F_l(U^{-1}z, \overline{U^{-1}z}, \lambda u) \neq 0.$$

The relation

$$\langle Uz, Uz \rangle = \lambda \langle z, z \rangle, \quad \lambda = \text{sign}\{\rho\}$$

yields

$$(2.31) \quad \lambda = \frac{1}{n} \Delta \langle Uz, Uz \rangle = \pm 1.$$

Then we take a value z, u in the equality (2.30) such that

$$F_l(z, \bar{z}, u) \in \mathbb{R} \setminus \{0\}$$

and define

$$\rho_1(U) = \left(\frac{\lambda F_l(U^{-1}z, \overline{U^{-1}z}, \lambda u)}{F_l(z, \bar{z}, u)} \right)^{\frac{2}{l-2}}.$$

By the unique factorization of a polynomial, we have

$$|\rho| = \rho_1(U)$$

regardless the choice of the value z, u . Hence, by the equality (2.31), we define

$$\rho(U) \equiv \frac{1}{n} \Delta \langle Uz, Uz \rangle \cdot \rho_1(U)$$

so that

$$(2.32) \quad \rho = \rho(U)$$

for all

$$(U, a, \rho, r) \in H(M).$$

Suppose that the real hypersurface $\phi_\sigma(M)$ is defined near the origin up to weight $l+1$ by the equation

$$\begin{aligned} v - \langle z, z \rangle &= \rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) + F_{l+1}^*(z, \bar{z}, u) + O(l+2) \\ &= F_l(z, \bar{z}, u) + F_{l+1}(z, \bar{z}, u) + O(l+2). \end{aligned}$$

By using the equality

$$\rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) = F_l(z, \bar{z}, u),$$

we obtain

$$F_{l+1}^*(z, \bar{z}, u) = H_{l+1}(z, \bar{z}, u; \rho^{-1}Ca) + \rho F_{l+1}(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u)$$

where $a^* \rightarrow H_{l+1}(z, \bar{z}, u; a^*)$ is the injective linear mapping in Lemma 2.8.

Then the following requirement

$$F_{l+1}^*(z, \bar{z}, u) = F_{l+1}(z, \bar{z}, u)$$

yields

$$H_{l+1}(z, \bar{z}, u; a^*) = F_{l+1}(z, \bar{z}, u) - \rho F_{l+1}(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u).$$

Then, by the equality (2.32), the equality

$$\begin{aligned} H_{l+1}(z, \bar{z}, u; a^*) &= F_{l+1}(z, \bar{z}, u) - \rho F_{l+1}(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) \\ &= F_{l+1}(z, \bar{z}, u) - \text{sign}\{\rho(U)\} |\rho(U)|^{\frac{l+3}{2}} F_{l+1}(U^{-1}z, \overline{U^{-1}z}, \text{sign}\{\rho(U)\} u), \end{aligned}$$

yields a unique function $a^*(U)$ of U satisfying

$$a^* = \rho^{-1}Ca = a^*(U).$$

Hence we obtain a unique function $a(U)$ of U such that

$$\begin{aligned} a &= a(U) \\ (2.33) \quad &\equiv \rho(U) |\rho(U)|^{-\frac{1}{2}} U^{-1} a^*(U) \end{aligned}$$

for all

$$(U, a, \rho, r) \in H(M).$$

Then we decompose the normalization ϕ_σ as follows:

$$\phi_\sigma = \phi_2 \circ \phi_1,$$

where ϕ_1, ϕ_2 are normalizations with the initial values σ_1, σ_2 respectively:

$$\sigma_1 = (id_{n \times n}, a, 1, 0) \quad \text{and} \quad \sigma_2 = (C, 0, \rho, r).$$

Suppose that the real hypersurface $\phi_1(M)$ is defined by the equation

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + \tilde{F}_{l+1}(z, \bar{z}, u) + \tilde{F}_{l+2}(z, \bar{z}, u) + O(l+3)$$

where the functions $\tilde{F}_{l+1}(z, \bar{z}, u)$ and $\tilde{F}_{l+2}(z, \bar{z}, u)$ depend of the parameter a , i.e.,

$$\begin{aligned} \tilde{F}_{l+1}(z, \bar{z}, u) &= \tilde{F}_{l+1}(z, \bar{z}, u; a) \\ \tilde{F}_{l+2}(z, \bar{z}, u) &= \tilde{F}_{l+2}(z, \bar{z}, u; a). \end{aligned}$$

Then suppose that the real hypersurface $\phi_\sigma(M)$ is defined near the origin up to weight $l+2$ by the equation

$$\begin{aligned} v &= \langle z, z \rangle + \rho F_l(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) + \rho \tilde{F}_{l+1}(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) \\ &\quad + \rho \tilde{F}_{l+2}(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) \\ &\quad - \frac{r}{2} \left\{ \sum_{\min(s,t) \geq 2} (l+s+t) u F_{st}(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) \right. \\ &\quad \quad + \sum_{\min(s,t) \geq 2} 2(s-t) i \langle z, z \rangle F_{st}(C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) \\ &\quad \quad \left. - \sum_{\min(s,t) \geq 2} 2\rho^{-1} \langle z, z \rangle^2 \left(\frac{\partial F_{st}}{\partial u} \right) (C^{-1}z, \overline{C^{-1}z}, \rho^{-1}u) \right\} \\ &\quad + O(l+3) \\ &= F_l(z, \bar{z}, u) + F_{l+1}(z, \bar{z}, u) + F_{l+2}(z, \bar{z}, u) + O(l+3), \end{aligned}$$

where

$$F_l(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u).$$

Hence we have the equality

$$\begin{aligned}
& -\frac{r}{2} \left\{ \sum_{\min(s,t) \geq 2} ((l+s+t)u + 2(s-t)i\langle z, z \rangle) F_{st}(z, \bar{z}, u) \right. \\
& \quad \left. - \sum_{\min(s,t) \geq 2} 2\langle z, z \rangle^2 \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \right\} \\
(2.34) \quad & = \rho^{-1} F_{l+2}(Cz, \overline{Cz}, \rho u) - \tilde{F}_{l+2}(z, \bar{z}, u; a).
\end{aligned}$$

Note that $F_l(z, \bar{z}, u) \neq 0$ implies

$$\sum_{\min(s,t) \geq 2} \left\{ ((l+s+t)u + 2(s-t)i\langle z, z \rangle) F_{st}(z, \bar{z}, u) - 2\langle z, z \rangle^2 \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \right\} \neq 0.$$

Otherwise, we would have

$$\begin{aligned}
& (l+s+t)u F_{st}(z, \bar{z}, u) + 2(s-t)i\langle z, z \rangle F_{s-1, t-1}(z, \bar{z}, u) \\
& - 2\langle z, z \rangle^2 \left(\frac{\partial F_{s-2, t-2}}{\partial u} \right) (z, \bar{z}, u) = 0
\end{aligned}$$

which yields

$$F_{st}(z, \bar{z}, u) = 0 \quad \text{for all } s, t.$$

From the equalities (2.32) and (2.33), we have

$$\rho = \rho(U), \quad a = a(U)$$

for all

$$(U, a, \rho, r) \in H(M).$$

Then we take a value z, u in the equality (2.34) such that

$$\sum_{\min(s,t) \geq 2} \left\{ ((l+s+t)u + 2(s-t)i\langle z, z \rangle) F_{st}(z, \bar{z}, u) - 2\langle z, z \rangle^2 \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \right\} \in \mathbb{R} \setminus \{0\}$$

and define

$$r(U) = \frac{-2 \left\{ \rho(U)^{-1} |\rho(U)|^{\frac{l+2}{2}} F_{l+2}(Uz, \overline{Uz}, \lambda u) - \tilde{F}_{l+2}(z, \bar{z}, u; a(U)) \right\}}{\sum_{\min(s,t) \geq 2} \left\{ ((l+s+t)u + 2(s-t)i\langle z, z \rangle) F_{st}(z, \bar{z}, u) - 2\langle z, z \rangle^2 \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \right\}}.$$

By the unique factorization of a polynomial, we have

$$r = r(U)$$

regardless the choice of the value z, u . Thus the equality (2.34) yields a unique function $r(U)$ of U satisfying

$$r = r(U)$$

for all

$$(U, a, \rho, r) \in H(M).$$

This completes the proof. \square

3. COMPACT LOCAL AUTOMORPHISM GROUPS

3.1. Compactness.

Lemma 3.1. *Let M be a nondegenerate analytic real hypersurface defined by*

$$v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0,$$

and ϕ_σ be a normalization of M with initial value $\sigma \in H$. Suppose that ϕ_σ transforms M to a real hypersurface defined by the equation

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u; \sigma).$$

Then the functions $\phi_\sigma(z, w)$ and $F^(z, \bar{z}, u; \sigma)$ are analytic of*

$$\sigma = (U, a, \rho, r) \in H.$$

Further, each coefficient

$$\left(\frac{\partial^{|I|+l} \phi_\sigma}{\partial z^{|I|} \partial w^l} \right)_0 \quad \text{and} \quad \left(\frac{\partial^{|I|+|J|+l} F^*}{\partial z^{|I|} \partial \bar{z}^{|J|} \partial u^l} \right)_0$$

depends polynomially on the parameters

$$C \equiv \sqrt{|\rho|}U, \quad C^{-1}, \quad \rho, \quad \rho^{-1}, \quad a, \quad r.$$

In the paper [Pa3], we have given the proof of Lemma 3.1.

Let M be a real hypersurface M in normal form. We define the isotropy subgroup $H(M)$ of M at the origin as follows:

$$H(M) = \{\sigma \in H : \phi_\sigma(M) = M\}$$

where ϕ_σ is a normalization of M with initial value $\sigma \in H$. By Lemma 3.1, the group H is homeomorphic to the set of germs ϕ_σ , $\sigma \in H$, with a topology induced from the natural compact-open topology. Further, by Lemma 2.10 and Lemma 3.1, the group $H(M)$ is isomorphic as Lie group to the local automorphism group of M .

Lemma 3.2. *Let M be a nonspherical analytic real hypersurface and $H(M)$ be the isotropy subgroup of M such that there is a real number $c \geq 1$ satisfying*

$$\sup_{(U, a, \rho, r) \in H(M)} \|U\| \leq c < \infty.$$

Then there exists a real number $e > 0$ satisfying

$$|a| \leq e, \quad e^{-1} \leq |\rho| \leq e, \quad |r| \leq e$$

for all elements

$$(U, a, \rho, r) \in H(M)$$

where e may depend on M and c .

Proof. For the parameter ρ , we have

$$\begin{aligned} |\rho(U)|^{\frac{l-2}{2}} &= \frac{|F_l(U^{-1}z, \overline{U^{-1}z}, \lambda u)|}{|F_l(z, \bar{z}, u)|} \\ |\rho(U)|^{-\frac{l-2}{2}} &= \frac{|F_l(Uz, \overline{Uz}, \lambda u)|}{|F_l(z, \bar{z}, u)|} \end{aligned}$$

whenever we take a value z, u satisfying

$$F_l(z, \bar{z}, u) \neq 0.$$

Hence we have the following estimate:

$$\begin{aligned} |\rho(U)|^{\frac{l-2}{2}} &\leq \sup_U \frac{|F_l(U^{-1}z, \overline{U^{-1}z}, \lambda u)|}{|F_l(z, \bar{z}, u)|} \\ |\rho(U)|^{-\frac{l-2}{2}} &\leq \sup_U \frac{|F_l(Uz, \overline{Uz}, \lambda u)|}{|F_l(z, \bar{z}, u)|} \end{aligned}$$

so that

$$\left(\sup_U \frac{|F_l(Uz, \overline{Uz}, \lambda u)|}{|F_l(z, \bar{z}, u)|} \right)^{-1} \leq |\rho(U)|^{\frac{l-2}{2}} \leq \sup_U \frac{|F_l(U^{-1}z, \overline{U^{-1}z}, \lambda u)|}{|F_l(z, \bar{z}, u)|}.$$

Note that there is a real number d depending only on $F_l(z, \bar{z}, u)$ such that

$$\frac{|F_l(U^{-1}z, \overline{U^{-1}z}, \lambda u)|}{|F_l(z, \bar{z}, u)|} \leq d_1 \cdot c^l$$

where

$$c \equiv \sup_{(U, a, \rho, r) \in H(M)} \|U\| \geq 1.$$

Thus we obtain

$$d_1^{-\frac{2}{l-2}} \cdot c^{-\frac{2l}{l-2}} \leq |\rho(U)| \leq d_1^{\frac{2}{l-2}} \cdot c^{\frac{2l}{l-2}}.$$

For the parameter a , we have

$$\begin{aligned} &H_{l+1}(z, \bar{z}, u; a^*(U)) \\ &= F_{l+1}(z, \bar{z}, u) - \text{sign}\{\rho(U)\} |\rho(U)|^{\frac{l+3}{2}} F_{l+1}(U^{-1}z, \overline{U^{-1}z}, \text{sign}\{\rho(U)\} u) \end{aligned}$$

where

$$a^*(U) = \rho(U)^{-1} \sqrt{|\rho(U)|} U a(U).$$

Since the mapping $a \mapsto H_{l+1}(z, \bar{z}, u; a)$ is injective and the function $H_{l+1}(z, \bar{z}, u; a)$ depends only on $F_l(z, \bar{z}, u)$, we have the following estimate:

$$|a^*(U)| \leq d_2^* \cdot c^{\frac{2(l^2+l-1)}{l-2}}$$

which yields

$$|a(U)| \leq d_2 \cdot c^{\frac{2(l^2+2l-2)}{l-2}}$$

where d_2^*, d_2 depend only on $F_l(z, \bar{z}, u)$ and $F_{l+1}(z, \bar{z}, u)$.

For the parameter r , we have

$$\begin{aligned} &-\frac{r(U)}{2} \left\{ \sum_{\min(s,t) \geq 2} (l+s+t) u F_{st}(z, \bar{z}, u) \right. \\ &\quad + \sum_{\min(s,t) \geq 2} 2(s-t) i \langle z, z \rangle F_{st}(z, \bar{z}, u) \\ &\quad \left. - \sum_{\min(s,t) \geq 2} 2 \langle z, z \rangle^2 \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \right\} \\ &= \lambda |\rho(U)|^{\frac{l}{2}} F_{l+2}(Uz, \overline{Uz}, \lambda u) - \tilde{F}_{l+2}(z, \bar{z}, u; a). \end{aligned}$$

By Lemma 3.1, the function $\tilde{F}_{l+2}(z, \bar{z}, u; a)$ depend polynomially, in fact, quadratically, on the parameter a . Hence we obtain the following estimate:

$$|r(U)| \leq d_3 \cdot c^L \quad \text{for some } L \in \mathbb{N}$$

where d_3 depends only on $F_l(z, \bar{z}, u)$, $F_{l+1}(z, \bar{z}, u)$, and $F_{l+2}(z, \bar{z}, u)$.

Then we take

$$e = \max \left\{ d_1^{\frac{2}{l-2}} \cdot c^{\frac{2l}{l-2}}, \quad d_2 \cdot c^{\frac{2(l^2+2l-2)}{l-2}}, \quad d_3 \cdot c^L \right\}.$$

This completes the proof. \square

Theorem 3.3. *Let M be a nonspherical analytic real hypersurface in normal form. Suppose that there is a real number $c \geq 1$ satisfying*

$$(3.1) \quad \sup_{(U, a, \rho, r) \in H(M)} \|U\| \leq c < \infty.$$

Then the group $H(M)$ is compact.

Proof. By Theorem 2.12, the group $H(M)$ is isomorphic to the following group:

$$\{U : (U, a, \rho, r) \in H(M)\}.$$

We claim that the group $H(M)$ is closed under the condition (3.1). Suppose that there is a convergent sequence in $GL(n; \mathbb{C})$ such that

$$U_m \in \{U : (U, a, \rho, r) \in H(M)\} \quad \text{for all } m \in \mathbb{N}$$

and, by the condition (3.1),

$$\lim_{m \rightarrow \infty} U_m = U \in GL(n; \mathbb{C}).$$

Then, by the functions $\rho(U), a(U), r(U)$ in Theorem 2.12, we have the following sequence:

$$(U_m, a(U_m), \rho(U_m), r(U_m)) \in H(M).$$

Under the condition (3.1), by Lemma 3.2, there is a real number $e > 0$ such that

$$|a(U_m)| \leq e, \quad e^{-1} \leq |\rho(U_m)| \leq e, \quad |r(U_m)| \leq e \quad \text{for all } m.$$

Then, by compactness, there is a subsequence m_j such that the following limits exists:

$$\begin{aligned} a &= \lim_{j \rightarrow \infty} a(U_{m_j}), \\ \rho &= \lim_{j \rightarrow \infty} \rho(U_{m_j}), \\ r &= \lim_{j \rightarrow \infty} r(U_{m_j}), \end{aligned}$$

which satisfy

$$|a| \leq e, \quad e^{-1} \leq |\rho| \leq e, \quad |r| \leq e.$$

Then we consider the following subset K of H given by

$$\begin{aligned} K &= \left\{ (U, a, \rho, r) \in H : \frac{1}{c} \leq \|U\| \leq c, \right. \\ &\quad \left. |a| \leq e, \quad e^{-1} \leq |\rho| \leq e, \quad |r| \leq e \right\}. \end{aligned}$$

Note that the set K is compact and

$$(U_m, a(U_m), \rho(U_m), r(U_m)) \in K \quad \text{for all } m.$$

Then, by Lemma 3.1, for each $\sigma \in K$, there exist real numbers $\varepsilon_\sigma, \delta_\sigma > 0$ such that all normalizations

$$\phi_{\sigma'}, \quad \sigma' \in K \cap \{\tau \in GL(n; \mathbb{C}) : \|\tau - \sigma\| \leq \varepsilon_\sigma\}$$

as a power series at the origin converge absolutely and uniformly on the open ball $B(0; \delta_\sigma)$. Notice that the following family of open sets

$$\{\tau \in GL(n+2; \mathbb{C}) : \|\tau - \sigma\| < \varepsilon_\sigma\}, \quad \sigma \in K$$

is an open covering of the set K . Since K is compact, there is a finite subcover, say,

$$\{\tau \in GL(n+2; \mathbb{C}) : \|\tau - \sigma_j\| < \varepsilon_{\sigma_j}\}, \quad \sigma_j \in H(M), \quad j = 1, \dots, l.$$

Then we set

$$\delta = \min_{1 \leq j \leq l} \{\delta_{\sigma_j}\} > 0$$

so that each normalization ϕ_σ , $\sigma \in K$, as a power series at the origin converges absolutely and uniformly on the open ball $B(0; \delta)$. Thus, by Motel theorem, the family of normalizations ϕ_σ , $\sigma \in K$, are a normal family on $B(0; \delta)$.

By a standard argument of a normal family, passing to a subsequence of $\{m_j\}$, if necessary, there is a holomorphic mapping ϕ on the open ball $B(0; \delta)$ such that

$$\phi = \lim_{j \rightarrow \infty} \phi_{\sigma_{m_j}}$$

where

$$\sigma_{m_j} = (U_{m_j}, a(U_{m_j}), \rho(U_{m_j}), r(U_{m_j})).$$

Then, for $\phi = (f, g)$, we have

$$\begin{aligned} \left(\frac{\partial f}{\partial z} \Big|_0 \right) &= \lim_{j \rightarrow \infty} \sqrt{|\rho(U_{m_j})|} U_{m_j} = \sqrt{|\rho|} U \\ \left(\frac{\partial f}{\partial w} \Big|_0 \right) &= - \lim_{j \rightarrow \infty} \sqrt{|\rho(U_{m_j})|} U_{m_j} a(U_{m_j}) = \sqrt{|\rho|} U a \\ \left(\frac{\partial g}{\partial w} \Big|_0 \right) &= \lim_{j \rightarrow \infty} \rho(U_{m_j}) = \rho \\ \left(\frac{\partial^2 g}{\partial w^2} \Big|_0 \right) &= 2 \lim_{j \rightarrow \infty} \rho(U_{m_j}) r(U_{m_j}) = 2\rho r. \end{aligned}$$

Note that

$$0 < |\det \phi'| = |\rho|^{\frac{n+2}{2}} |\det U| < \infty.$$

Thus, by Hurwitz theorem, the mapping ϕ is a biholomorphic mapping on the ball $B(0; \delta)$. Further, notice

$$\phi_{\sigma_m}(M \cap B(0; \delta)) \subset M \quad \text{for all } m \in \mathbb{N},$$

so that

$$\phi(M \cap B(0; \delta)) \subset M.$$

Hence the mapping ϕ is a biholomorphic automorphism of M with initial value $\sigma \in H$ such that

$$\sigma = (U, a, \rho, r) \in H(M).$$

Thus we have showed

$$U = \lim_{m \rightarrow \infty} U_m \in \{U : (U, a, \rho, r) \in H(M)\}.$$

Then the group

$$\{U : (U, a, \rho, r) \in H(M)\} \subset GL(n+2; \mathbb{C})$$

is closed so that it is a compact Lie group. Therefore, we prove our claim that the group $H(M)$ is closed. Hence $H(M)$ is a compact Lie group. This completes the proof. \square

3.2. Theorem of a germ of a biholomorphic mapping. We study the analytic continuation of a germ of a biholomorphic mapping to a finite neighborhood(cf. [Vi]).

Lemma 3.4. *Let M be a nonspherical analytic real hypersurface in normal form and $H(M)$ be the isotropy subgroup of M such that there is a real number $c \geq 1$ satisfying*

$$\sup_{(U, a, \rho, r) \in H(M)} \|U\| \leq c < \infty.$$

Then there is a real number $\delta > 0$ such that all local automorphisms of M , ϕ_σ , $\sigma \in H(M)$, converge absolutely and uniformly on the open ball $B(0; \delta)$.

Proof. By Lemma 3.1, for each $\sigma \in H(M)$, there exist real numbers $\varepsilon_\sigma, \delta_\sigma > 0$ such that all normalizations

$$\phi_{\sigma'}, \quad \sigma' \in H(M) \cap \{\tau \in GL(n+2; \mathbb{C}) : \|\tau - \sigma\| \leq \varepsilon_\sigma\}$$

as a power series at the origin converges absolutely and uniformly on the open ball $B(0; \delta_\sigma)$.

Note that the following family

$$\{\tau \in GL(n+2; \mathbb{C}) : \|\tau - \sigma\| < \varepsilon_\sigma\}, \quad \sigma \in H(M)$$

is an open covering of the set $H(M)$. By Lemma 3.3, $H(M)$ is compact. Thus there is a finite subcover, say,

$$\{\tau \in GL(n+2; \mathbb{C}) : \|\tau - \sigma_j\| < \varepsilon_{\sigma_j}\}, \quad \sigma_j \in H(M), \quad j = 1, \dots, m.$$

Then we take

$$\delta = \min_{1 \leq j \leq m} \{\delta_{\sigma_j}\} > 0.$$

This completes the proof. \square

Theorem 3.5 (Vitushkin). *Let M, M' be a nonspherical analytic real hypersurface and p, p' be points respectively of M, M' such that the two germs M at p and M' at p' are biholomorphically equivalent. Suppose that there is a real number $c \geq 1$ satisfying*

$$\sup_{(U, a, \rho, r) \in H_p(M)} \|U\| \leq c < \infty$$

where $H_p(M)$ is a local automorphism group of M at the point p in a normal coordinate. Then there is a real number $\delta > 0$ depending only on M and M' such that each biholomorphic mapping ϕ of M near the point p is analytically continued

to the open ball $B(p; \delta)$ whenever $\phi(p) = p'$ and there is an open neighborhood $U \subset B(p; \delta)$ of the point p satisfying

$$\phi(U \cap M) \subset M'.$$

Proof. We take a biholomorphic mapping ϕ of M to M' such that

$$\phi(p) = p'$$

and, for an open neighborhood U of the point p ,

$$\phi(U \cap M) \subset M'.$$

Then we take normalizations ϕ_1, ϕ_2 respectively of M, M' such that ϕ_1, ϕ_2 translate the points p, p' to the origin and there exist open neighborhoods U_1, U_2 respectively of p, p' and a real hypersurface M^* in normal form satisfying

$$\begin{aligned} \phi_1(U_1 \cap M) &\subset M^* \\ \phi_2(U_2 \cap M) &\subset M^*. \end{aligned}$$

Then, we obtain a biholomorphic mapping ϕ^* defined by

$$\phi^* = \phi_2 \circ \phi \circ \phi_1^{-1}.$$

Notice that the mapping ϕ^* is a local automorphism of M^* . By Lemma 3.4, there is a real number $\delta^* > 0$ such that the mapping ϕ^* continues holomorphically to the open ball $B(0; \delta^*)$ satisfying

$$\begin{aligned} B(0; \delta^*) &\subset \phi_1(U_1) \\ B(0; \delta^*) &\subset \phi_2(U_2). \end{aligned}$$

Then the mapping

$$\phi = \phi_2^{-1} \circ \phi_{\sigma^*} \circ \phi_1$$

is biholomorphically continued to the open set

$$U_1 \cap \phi_1^{-1}(B(0; \delta^*)).$$

We take a real number $\delta > 0$ such that

$$B(p; \delta) \subset U_1 \cap \phi_1^{-1}(B(0; \delta^*)).$$

This completes the proof. \square

3.3. Kruzhilin-Loboda Theorem. By Lemma 2.10, we have a H -group action on real hypersurfaces in normal form. Then the orbit structure in normal form may be studied by examining the isotropy subgroup $H(M)$ for a real hypersurface M in normal form.

Lemma 3.6. *Let K be a subset of H . The necessary and sufficient condition for the set K to be conjugate to a subset of*

$$\left\{ (U, 0, \pm 1, 0) \equiv \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H \right\}$$

is given as follows:

$$K \subset \{(U, a, \pm 1, r) \in H\}$$

and there exist a vector $d \in \mathbb{C}^n$ and a real number $e \in \mathbb{R}$ such that

$$\begin{aligned} (id_{n \times n} - \lambda U^{-1})d &= a \\ (1 - \lambda)e + i\langle d, a \rangle - i\langle a, d \rangle &= r \end{aligned}$$

for all

$$(U, a, \lambda, r) \in K.$$

Proof. Each element of H is decomposed as follows:

$$\begin{pmatrix} \rho' & 0 & 0 \\ -C'a' & C' & 0 \\ -r' - i\langle a', a' \rangle & 2ia'^\dagger & 1 \end{pmatrix} = \begin{pmatrix} \rho' & 0 & 0 \\ 0 & C' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a' & id_{n \times n} & 0 \\ -r' - i\langle a', a' \rangle & 2ia'^\dagger & 1 \end{pmatrix}$$

where

$$a'^\dagger z = \langle z, a' \rangle.$$

Note that

$$\begin{pmatrix} \rho' & 0 & 0 \\ 0 & C' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho' & 0 & 0 \\ 0 & C' & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & C'UC'^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the straight forward computation yields

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 \\ -a' & id_{n \times n} & 0 \\ -r' - i\langle a', a' \rangle & 2ia'^\dagger & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 & 0 \\ -Ca & C & 0 \\ -r - i\langle a, a \rangle & 2ia^\dagger & 1 \end{pmatrix} \times \\ & \begin{pmatrix} 1 & 0 & 0 \\ -a' & id_{n \times n} & 0 \\ -r' - i\langle a', a' \rangle & 2ia'^\dagger & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \rho & 0 & 0 \\ -Ca^* & C & 0 \\ -r^* - i\langle a^*, a^* \rangle & 2ia^{*\dagger} & 1 \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} a^* &= \rho C^{-1}a' + a - a' \\ r^* &= \rho r' - r' + r + i\langle C(a - a'), a' \rangle - i\langle a', C(a - a') \rangle \\ &\quad + i\langle a, a' \rangle - i\langle a', a \rangle. \end{aligned}$$

Hence the necessary and sufficient condition for the set K to be conjugate to a subset of

$$\left\{ (U, 0, \pm 1, 0) \equiv \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H \right\}$$

is given by

$$|\rho| = 1 \quad \text{and} \quad a^* = r^* = 0$$

for all

$$(U, a, \rho, r) \in K.$$

The equalities $a^* = r^* = 0$ yields

$$\begin{aligned} (id_{n \times n} - \rho C^{-1})a' &= a \\ (1 - \rho)r' + i\langle a', a \rangle - i\langle a, a' \rangle &= r. \end{aligned}$$

The necessary and sufficient condition is equivalent to the existence of a vector $a' \in \mathbb{C}^n$ and $r' \in \mathbb{R}$ satisfying

$$\begin{aligned} |\rho| &= 1 \\ (id_{n \times n} - \rho C^{-1})a' &= a \\ (1 - \rho)r' + i\langle a', a \rangle - i\langle a, a' \rangle &= r. \end{aligned}$$

for all

$$(U, a, \rho, r) \in K.$$

This completes the proof. \square

Theorem 3.7 (Kruzhilin-Loboda). *Let M be a real hypersurface in normal form and $H(M)$ be the isotropy group of M such that there is a real number $c \geq 1$ satisfying*

$$\sup_{(U, a, \rho, r) \in H(M)} \|U\| \leq c < \infty.$$

Then there exists an element $\sigma \in H$ satisfying

$$\sigma H(M) \sigma^{-1} \subset \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H \right\}.$$

Proof. By Lemma 3.3, the group

$$G = \{U : (U, a, \rho, r) \in H(M)\},$$

is a compact Lie group. Thus we have a unique Haar measure μ on G such that

$$\int_{V \in G} d\mu(V) = 1.$$

Suppose that M is defined by the equation

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + F_{l+1}(z, \bar{z}, u) + O(l+2).$$

By Theorem 2.12, there is a function $\rho(U)$ satisfying

$$\rho = \rho(U)$$

for all

$$(U, a, \rho, r) \in H(M).$$

Then we have the following identity

$$|\rho(U)|^{\frac{l-2}{2}} F_l(z, \bar{z}, u) = \text{sign}\{\rho(U)\} F_l\left(U^{-1}z, \overline{U^{-1}z}, \text{sign}\{\rho(U)\}u\right)$$

which yields

$$F_l(z, \bar{z}, u) = \frac{\int_G \left\{ \text{sign}\{\rho(V)\} F_l\left(V^{-1}z, \overline{V^{-1}z}, \text{sign}\{\rho(V)\}u\right) \right\} d\mu(V)}{\int_G |\rho(V)|^{\frac{l-2}{2}} d\mu(V)}.$$

Hence we easily see

$$\text{sign}\{\rho(U)\} F_l \left(U^{-1}z, \overline{U^{-1}z}, \text{sign}\{\rho(U)\}u \right) = F_l(z, \bar{z}, u)$$

so that

$$|\rho(U)|^{\frac{l-2}{2}} F_l(z, \bar{z}, u) = F_l(z, \bar{z}, u).$$

Thus we have

$$|\rho(U)| \equiv 1 \quad \text{for all } U \in G$$

so that

$$H(M) \subset \{(U, a, \pm 1, r) \in H\}.$$

By Theorem 2.12, there is a function $a(U)$ satisfying

$$a = a(U)$$

for all

$$(U, a, \pm 1, r) \in H(M).$$

Then we have the identity

$$\begin{aligned} & H_{l+1}(z, \bar{z}, u; \text{sign}\{\rho(U)\}Ua(U)) \\ &= F_{l+1}(z, \bar{z}, u) - \text{sign}\{\rho(U)\} F_{l+1} \left(U^{-1}z, \overline{U^{-1}z}, \text{sign}\{\rho(U)\}u \right). \end{aligned}$$

Hence there is a vector a^* satisfying

$$\begin{aligned} & H_{l+1}(z, \bar{z}, u; a^*) \\ &= F_{l+1}(z, \bar{z}, u) - \int_G \left\{ \text{sign}\{\rho(V)\} F_{l+1} \left(V^{-1}z, \overline{V^{-1}z}, \text{sign}\{\rho(V)\}u \right) \right\} d\mu(V) \end{aligned}$$

where

$$a^* = \int_G \text{sign}\{\rho(V)\} Va(V) d\mu(V).$$

Suppose that the normalization ϕ of M with initial value

$$(id_{n \times n}, -a^*, 1, 0) \in H$$

transforms M to a real hypersurface M' . Then M' is defined up to weight $l+1$ by the equation

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + F_{l+1}^*(z, \bar{z}, u) + O(l+2)$$

where

$$F_{l+1}^*(z, \bar{z}, u) = \int_G \left\{ \text{sign}\{\rho(V)\} F_{l+1} \left(V^{-1}z, \overline{V^{-1}z}, \text{sign}\{\rho(V)\}u \right) \right\} d\mu(V).$$

We easily see that

$$\text{sign}\{\rho(U)\} F_{l+1}^* \left(U^{-1}z, \overline{U^{-1}z}, \text{sign}\{\rho(U)\}u \right) = F_{l+1}^*(z, \bar{z}, u).$$

Because the linear mapping $a^* \mapsto H_{l+1}(z, \bar{z}, u; a^*)$ is injective, we obtain

$$H(M') \subset \{(U, 0, \pm 1, r) \in H\}.$$

Suppose that M' is defined up to weight $l+2$ by the equation

$$F(M') = F_l(z, \bar{z}, u) + F_{l+1}^*(z, \bar{z}, u) + F_{l+2}(z, \bar{z}, u) + O(l+3).$$

By Theorem 2.12, there is a function $r(U)$ satisfying

$$r = r(U)$$

for all

$$(U, 0, \pm 1, r) \in H(M').$$

Then we have the following identity

$$\begin{aligned} & -\frac{r(U)}{2} \left\{ \sum_{\min(s,t) \geq 2} (l+s+t)uF_{st}(z, \bar{z}, u) \right. \\ & \quad + \sum_{\min(s,t) \geq 2} 2(s-t)i\langle z, z \rangle F_{st}(z, \bar{z}, u) \\ & \quad \left. - \sum_{\min(s,t) \geq 2} 2\langle z, z \rangle^2 \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \right\} \\ & = \text{sign}\{\rho(U)\} F_{l+2}(Uz, \overline{Uz}, \text{sign}\{\rho(U)\}u) - F_{l+2}(z, \bar{z}, u) \end{aligned}$$

where

$$F_l(z, \bar{z}, u) = \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u).$$

Hence there is a real number r^* satisfying

$$\begin{aligned} & -\frac{r^*}{2} \left\{ \sum_{\min(s,t) \geq 2} (l+s+t)uF_{st}(z, \bar{z}, u) \right. \\ & \quad + \sum_{\min(s,t) \geq 2} 2(s-t)i\langle z, z \rangle F_{st}(z, \bar{z}, u) \\ & \quad \left. - \sum_{\min(s,t) \geq 2} 2\langle z, z \rangle^2 \left(\frac{\partial F_{st}}{\partial u} \right) (z, \bar{z}, u) \right\} \\ & = \int_G \text{sign}\{\rho(V)\} F_{l+2}(Vz, \overline{Vz}, \text{sign}\{\rho(V)\}u) d\mu(V) - F_{l+2}(z, \bar{z}, u) \end{aligned}$$

where

$$r^* = \int_G r(V) d\mu(V).$$

Suppose that the normalization ϕ' of M' with initial value

$$(id, 0, 1, r^*) \in H$$

transforms M' to a real hypersurface M'' . Then M'' is defined up to weight $l+2$ by the equation

$$v = \langle z, z \rangle + F_l(z, \bar{z}, u) + F_{l+1}^*(z, \bar{z}, u) + F_{l+2}^*(z, \bar{z}, u) + O(l+3)$$

where

$$F_{l+2}^*(z, \bar{z}, u) = \int_G \text{sign}\{\rho(V)\} F_{l+2}(Vz, \overline{Vz}, \text{sign}\{\rho(V)\}u) d\mu(V).$$

We easily see that

$$\text{sign}\{\rho(U)\} F_{l+2}^*(Uz, \overline{Uz}, \text{sign}\{\rho(U)\}u) = F_{l+2}^*(z, \bar{z}, u)$$

which yields

$$H(M'') \subset \{(U, 0, \pm 1, 0) \in H\}.$$

Then we take a normalization ϕ_σ with initial value $\sigma \in H$ such that

$$\phi_\sigma(M) = M''.$$

Then, by Lemma 2.10, we obtain

$$\sigma H(M) \sigma^{-1} = H(M'').$$

This completes the proof. \square

4. ANALYTIC CONTINUATION OF A NORMALIZING MAPPING

4.1. Chains on a spherical real hypersurface. By Theorem 0.4, each biholomorphic automorphism of the real hyperquadric

$$v = \langle z, z \rangle$$

is uniquely given by a composition of an affine mapping

$$(4.1) \quad \begin{aligned} z^* &= z + b \\ w^* &= w + 2i\langle z, b \rangle + c + i\langle b, b \rangle \end{aligned}$$

and a fractional linear mapping:

$$(4.2) \quad \phi = \phi_\sigma : \begin{cases} z^* = \frac{C(z-aw)}{1+2i\langle z, a \rangle - w(r+i\langle a, a \rangle)} \\ w^* = \frac{\rho w}{1+2i\langle z, a \rangle - w(r+i\langle a, a \rangle)} \end{cases}$$

where

$$b \in \mathbb{C}^n, \quad c \in \mathbb{R}$$

and the constants $\sigma = (C, a, \rho, r)$ satisfy

$$\begin{aligned} a &\in \mathbb{C}^n, \quad \rho \neq 0, \quad \rho, r \in \mathbb{R}, \\ C &\in GL(n; \mathbb{C}), \quad \langle Cz, Cz \rangle = \rho \langle z, z \rangle. \end{aligned}$$

Note that the local automorphism ϕ decomposes to

$$\phi = \varphi \circ \psi,$$

where

$$\psi : \begin{cases} z^* = \frac{z-aw}{1+2i\langle z, a \rangle - i\langle a, a \rangle w} \\ w^* = \frac{w}{1+2i\langle z, a \rangle - i\langle a, a \rangle w} \end{cases} \quad \text{and} \quad \varphi : \begin{cases} z^* = \frac{Cz}{1-rw} \\ w^* = \frac{\rho w}{1-rw} \end{cases}.$$

Lemma 4.1. *Let M be the real hyperquadric $v = \langle z, z \rangle$. Then the intersection of the real hyperquadric M by a complex line l is given by a point, a curve γ , or the complex line l itself. If the intersection is a curve γ , then γ is transversal to the complex tangent hyperplane at every point of γ .*

Proof. Let $(\kappa, \chi) \in \mathbb{C}^n \times \mathbb{C}$ be a point of the real hyperquadric $v = \langle z, z \rangle$ such that

$$\Im \chi = \langle \kappa, \kappa \rangle.$$

Then a complex line l passing through the point (κ, χ) is given by

$$\{(\kappa, \chi) + e(\mu, \nu) : e \in \mathbb{C}\}$$

for some nonzero vector $(\mu, \nu) \in \mathbb{C}^n \times \mathbb{C}$. Then the affine mapping (4.1) send the complex line l to another complex line l^* given by

$$\{(\kappa + b, \chi + 2i\langle \kappa, b \rangle + c + i\langle b, b \rangle) + e(\mu, \nu + 2i\langle \mu, b \rangle) : e \in \mathbb{C}\}.$$

Note that the complex line l^* passes through the origin by taking

$$b = -\kappa, \quad c = -\Re \chi.$$

Thus we reduce the discussion to complex lines passing through the origin.

Suppose that the complex line l is tangent to the complex tangent hyperplane at the origin so that l is given by

$$\{c(a, 0) : c \in \mathbb{C}\}$$

for some nonzero vector $a \in \mathbb{C}^n$. Then each point in the intersection of the real hyperquadric M by the complex line l satisfies

$$c\bar{c}\langle a, a \rangle = 0.$$

Thus we obtain that, whenever $\langle a, a \rangle \neq 0$,

$$M \cap \{c(a, 0) : c \in \mathbb{C}\} = \{(0, 0)\}$$

and, whenever $\langle a, a \rangle = 0$,

$$M \cap \{c(a, 0) : c \in \mathbb{C}\} = \{c(a, 0) : c \in \mathbb{C}\}.$$

Suppose that the complex line l is transversal to the complex tangent hyperplane at the origin so that l is given by

$$\{c(a, 1) : c \in \mathbb{C}\}$$

for some vector $a \in \mathbb{C}^n$. We claim that the complex tangent hyperplanes of the real hyperquadric M and the complex line l are transversal at each point of the intersection γ :

$$\gamma = M \cap \{c(a, 1) : c \in \mathbb{C}\}.$$

Let (ca, c) , $c \neq 0$, be a point of M so that

$$(4.3) \quad \frac{1}{2i}(c + \bar{c}) = c\bar{c}\langle a, a \rangle$$

and $(\mu, \nu) \in \mathbb{C}^n \times \mathbb{C}$ be a vector tangent to the complex tangent hyperplane of M at the point (ca, c) . Then we obtain

$$\nu - 2i\langle \mu, ca \rangle = 0$$

so that

$$\begin{aligned} (\mu, \nu) &= \mu_1(1, 0, \dots, 0, 0, 2ie_1\bar{c}\bar{a}^1) \\ &\quad + \mu_2(0, 1, \dots, 0, 0, 2ie_2\bar{c}\bar{a}^2) \\ &\quad + \dots \\ &\quad + \mu_n(0, 0, \dots, 0, 1, 2ie_n\bar{c}\bar{a}^n) \end{aligned}$$

where

$$\begin{aligned} \langle z, z \rangle &= e_1 z^1 \bar{z}^1 + \dots + e_n z^n \bar{z}^n \\ e_1, \dots, e_n &= \pm 1. \end{aligned}$$

Thus the transversality at γ is determined by the value:

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 & 2ie_1\overline{ca}^1 \\ 0 & 1 & \ddots & \vdots & 2ie_2\overline{ca}^2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 2ie_n\overline{ca}^n \\ a^1 & a^2 & \cdots & a^n & 1 \end{pmatrix} = 1 - 2i\overline{c}\langle a, a \rangle.$$

Suppose that

$$1 - 2i\overline{c}\langle a, a \rangle = 0.$$

Then the equality (4.3) yields $c = 0$. This is a contradiction to $c \neq 0$.

Thus the complex tangent hyperplanes of the real hyperquadric M and the complex line l are transversal at each point of the intersection γ . Therefore, the intersection γ is a curve transversal to the complex tangent hyperplanes of M at each point of γ . This completes the proof. \square

Let M be a nondegenerate analytic real hypersurface in a complex manifold and p be a point of M . Then we can take a normal coordinate with the center at the point p so that M is defined by

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0.$$

A connected open curve γ on M is called a chain if it is locally putted into the u -curve of a normal coordinate at each point of γ . A connected closed subarc of a chain γ shall be called a chain-segment.

Lemma 4.2. *Let M, M' be nondegenerate analytic real hypersurfaces and ϕ be a biholomorphic mapping on an open neighborhood U of a point $p \in M$ such that*

$$\phi(U \cap M) \subset M'.$$

Suppose that there is a chain γ of M passing through the point p . Then the analytic curve $\phi(U \cap \gamma)$ is a chain of M' .

Proof. Let $q \in \phi(U \cap \gamma)$. Since γ is a chain of M , there exist an open neighborhood V of the point $\phi^{-1}(q) \in \gamma$ and a normalization ϕ_1 of M such that ϕ_1 is biholomorphic V and

$$\phi_1(V \cap \gamma) \subset \{z = v = 0\}.$$

Note that $\phi_1 \circ \phi^{-1}$ is a normalization of M' such that, for a sufficiently small open neighborhood O of the point q , $\phi_1 \circ \phi^{-1}$ is biholomorphic on O and

$$\phi_1 \circ \phi^{-1}(O \cap \phi(V \cap \gamma)) \subset \{z = v = 0\}.$$

Since q is an arbitrary point of $\phi(U \cap \gamma)$, the analytic curve $\phi(U \cap \gamma)$ is a chain. This completes the proof. \square

Lemma 4.3. *Let M be a real hyperquadric $v = \langle z, z \rangle$ and $p = (\kappa, \chi) \neq 0$ be a point of M . Then there is a chain-segment $\gamma : [0, 1] \rightarrow M$ such that*

$$\gamma(0) = 0, \quad \gamma(1) = p$$

whenever

$$\Re \chi \neq 0 \quad \text{or} \quad \langle \kappa, \kappa \rangle \neq 0.$$

Proof. Let ϕ_σ be a local automorphism of a real hyperquadric with initial value $\sigma \in H$. Then the inverse ϕ_σ^{-1} of the local automorphism ϕ_σ is given by

$$(4.4) \quad \phi_\sigma^{-1} : \begin{cases} z = \frac{C^{-1}(z^* + \rho^{-1}Ca w^*)}{1 - 2i\langle z^*, \rho^{-1}Ca \rangle - w^*(-r\rho^{-1} + i\langle \rho^{-1}Ca, \rho^{-1}Ca \rangle)} \\ w = \frac{\rho^{-1}w^*}{1 - 2i\langle z^*, \rho^{-1}Ca \rangle - w^*(-r\rho^{-1} + i\langle \rho^{-1}Ca, \rho^{-1}Ca \rangle)} \end{cases}.$$

Thus the chain passing through the origin and transversal to the complex tangent hyperplane at the origin, γ , is given with a normal parametrization by

$$\gamma = \phi^{-1}(z^* = v^* = 0) : \begin{cases} z = \frac{\rho^{-1}au^*}{1 - \rho^{-1}u^*(-r + i\langle a, a \rangle)} \\ w = \frac{\rho^{-1}u^*}{1 - \rho^{-1}u^*(-r + i\langle a, a \rangle)} \end{cases}.$$

By taking $r = 0$, we easily see that the chain γ is the intersection of M and the complex line

$$\{c(a, 1) : c \in \mathbb{C}\}.$$

By Lemma 4.1, the chain γ is transversal to the complex tangent hyperplanes of M at each point of γ .

Let $(\kappa, \chi) \neq 0$ be a point in $\mathbb{C}^n \times \mathbb{C}$ on M such that

$$\Im \chi = \langle \kappa, \kappa \rangle.$$

Then we have $\chi \neq 0$ whenever

$$\Re \chi \neq 0 \quad \text{or} \quad \langle \kappa, \kappa \rangle \neq 0.$$

Note that the origin and the point (κ, χ) , $\chi \neq 0$, is connected by the chain

$$\Gamma = M \cap \{c(\chi^{-1}\kappa, 1) : c \in \mathbb{C}\}.$$

This completes the proof. \square

Lemma 4.4. *Let M be a spherical analytic real hypersurface. Then M is locally biholomorphic to a real hyperquadric.*

In the paper [Pa3], we have proved Lemma 4.4.

Theorem 4.5. *Let M be a spherical analytic real hypersurface and $\gamma : [0, 1] \rightarrow M$ be a curve such that $\gamma[0, \tau]$ is a chain-segment for each $\tau < 1$. Then $\gamma[0, 1]$ is a chain-segment of M .*

Proof. By Lemma 4.4, the real hypersurface M is biholomorphic to a real hyperquadric at the point $\gamma(1)$. Then, by Lemma 4.4, taking a normal coordinate with center at the point $\gamma(1)$ yields

$$v = \langle z, z \rangle$$

where the curve $\gamma[0, 1]$ touches the origin and the part $\gamma(0, 1)$ is a chain. By Lemma 4.1 and Lemma 4.3, there exist a chain Γ passing through the origin, an

open neighborhood U of the origin, and a normalization ϕ of M such that ϕ is biholomorphic on U and

$$\phi(\Gamma \cap U) \subset \{z = v = 0\}.$$

Since $\gamma[0, 1] \subset \Gamma$ and Γ is a chain of M , $\gamma[0, 1]$ is a chain-segment. This completes the proof. \square

Proposition 4.6. *Let M be a spherical analytic real hypersurface and p be a point of M . Suppose that there are an open cone V_θ with its vertex at the point p and euclidean angle θ , $0 < \theta < \frac{\pi}{2}$, to the complex tangent hyperplane at the point p , and an open neighborhood U of the point p . Then there is a number $\delta > 0$ such that, for each given curve $\eta : [0, 1] \rightarrow V_\theta \cap B(p; \delta)$, there is a continuous family of chain-segments*

$$\gamma : [0, 1] \times [0, 1] \rightarrow U \cap M$$

where $\gamma(s, \cdot) : [0, 1] \rightarrow U \cap M$ is a chain-segment of M for each $s \in [0, 1]$ satisfying

$$\gamma(s, 0) = p \quad \text{and} \quad \gamma(s, 1) = \eta(s) \quad \text{for all } s \in [0, 1].$$

Proof. By Lemma 4.4, there is a biholomorphic mapping ϕ of M near the point p to a real hyperquadric $v = \langle z, z \rangle$ such that $N(p) = 0$. Then we take a sufficiently small number $\varepsilon > 0$ so that each point $q \in \phi(V_\theta) \cap B(0; \varepsilon)$ is connected by a chain-segment $\gamma \subset \phi(U \cap M)$ to the origin. Then we take a number $\delta > 0$ such that

$$\phi(V_\theta \cap B(p; \delta)) \subset \phi(V_\theta) \cap B(0; \varepsilon).$$

By Lemma 4.3, there is a continuous family of complex line l_τ such that the intersection $l_\tau \cap \phi(U \cap M)$ is a chain and the chain $l_\tau \cap \phi(U \cap M)$ connects the point p and the point $\phi(\eta(\tau))$ for each $\tau \in [0, 1]$. Hence there is a continuous family of chain-segments

$$\Gamma : [0, 1] \times [0, 1] \rightarrow \phi(U \cap M)$$

where $\Gamma(\tau, \cdot) : [0, 1] \rightarrow \phi(U \cap M)$ is a chain-segment for each $\tau \in [0, 1]$ satisfying

$$\Gamma(\tau, 0) = 0 \quad \text{and} \quad \Gamma(\tau, 1) = \phi(\eta(\tau)) \quad \text{for all } \tau \in [0, 1].$$

Then, by Lemma 4.2, the desired family of chain-segments on M is given by

$$\gamma \equiv \phi^{-1} \circ \Gamma : [0, 1] \times [0, 1] \rightarrow U \cap M.$$

This completes the proof. \square

4.2. Chains on a nonspherical real hypersurface.

Lemma 4.7. *Let M be a nondegenerate analytic real hypersurface and p be a point of M . Suppose that there are an open cone V_θ with its vertex at the point p and euclidean angle θ , $0 < \theta < \frac{\pi}{2}$, to the complex tangent hyperplane at the point p , and an open neighborhood U of the point p . Then there is a number $\delta > 0$ such that, for each given curve $\eta : [0, 1] \rightarrow V_\theta \cap B(p; \delta)$, there is a continuous family of chain-segments*

$$\gamma : [0, 1] \times [0, 1] \rightarrow U \cap M$$

where $\gamma(s, \cdot) : [0, 1] \rightarrow U \cap M$ is a chain-segment of M for each $s \in [0, 1]$ satisfying

$$\gamma(s, 0) = p \quad \text{and} \quad \gamma(s, 1) = \eta(s) \quad \text{for all } s \in [0, 1].$$

Proof. By translation and unitary transformation, if necessary, we may assume that the point p is at the origin and the real hypersurface M is defined near the origin by

$$v = F(z, \bar{z}, u), \quad F|_0 = F_z|_0 = F_{\bar{z}}|_0 = 0$$

so that

$$F(z, \bar{z}, u) = \sum_{s=2}^{\infty} F_s(z, \bar{z}, u).$$

With a sufficiently small number $\varepsilon > 0$, we consider an analytic family of real hypersurfaces M_μ , $|\mu| \leq \varepsilon$, defined near the origin by the equations:

$$v = F^*(z, \bar{z}, u; \mu)$$

where

$$F^*(z, \bar{z}, u; \mu) = \sum_{s=2}^{\infty} \mu^{s-2} F_s(z, \bar{z}, u).$$

Note that the function $F^*(z, \bar{z}, u; \mu)$ is analytic of z, u, μ and the real hypersurface M_0 (i.e., $\mu = 0$) is spherical.

Then we obtain an analytic family of ordinary differential equations

$$(4.5) \quad p'' = Q(\tau, p, \bar{p}, p', \bar{p}'; \mu)$$

so that each chain γ passing through the origin on M_μ is given by

$$\gamma : \begin{cases} z = p(\tau) \\ w = \tau + iF^*(p(\tau), \bar{p}(\tau), \tau; \mu) \end{cases}$$

where $p(\tau)$ is a solution of the equation (4.5). The solution p of the equation (4.5) is given as an analytic function of τ, μ, a , where

$$a = p'(0).$$

In fact, for a given real number $\nu \in \mathbb{R}^+$, there are real numbers τ_1, ε_1 such that the analytic function

$$p = p(\tau, \mu, a)$$

converges absolutely and uniformly on the range

$$|a| \leq \nu, \quad |\tau| \leq \tau_1, \quad |\mu| \leq \varepsilon_1.$$

Since M_0 is spherical, by Theorem 4.6, for an open neighborhood U_0 of the origin and an open cone V_{θ_0} with its vertex at the origin and euclidean angle θ_0 , $0 < \theta_0 < \frac{\pi}{2}$, to the complex tangent hyperplane at the origin, there is a number $\delta_0 > 0$ such that, for each given curve $\eta : [0, 1] \rightarrow V_{\theta_0} \cap B(0; \delta_0)$, there is a continuous family of chain-segments

$$\gamma_0 : [0, 1] \times [0, 1] \rightarrow U_0 \cap M_0$$

where $\gamma_0(s, \cdot) : [0, 1] \rightarrow U_0 \cap M_0$ is a chain-segment of M_0 for each $s \in [0, 1]$ satisfying

$$\gamma_0(s, 0) = 0 \quad \text{and} \quad \gamma_0(s, 1) = \eta(s) \quad \text{for all } s \in [0, 1].$$

Then, for an open neighborhood U_1 of the origin and an open cone V_{θ_1} with its vertex at the origin and euclidean angle θ_1 , $0 < \theta_1 < \frac{\pi}{2}$, to the complex tangent

hyperplane at the origin, there exist real numbers $\mu_1, \delta_1 > 0$ such that, for each given curve $\eta : [0, 1] \rightarrow V_{\theta_1} \cap B(0; \delta_1)$, there is a continuous family of chain-segments

$$\gamma_1 : [0, 1] \times [0, 1] \rightarrow U_1 \cap M_{\mu_1}$$

where $\gamma_1(s, \cdot) : [0, 1] \rightarrow U_1 \cap M_{\mu_1}$ is a chain-segment of M_{μ_1} for each $s \in [0, 1]$ satisfying

$$\gamma_1(s, 0) = 0 \quad \text{and} \quad \gamma_1(s, 1) = \eta(s) \quad \text{for all } s \in [0, 1].$$

By the way, the real hypersurface M_μ , $\mu \neq 0$, is obtained from M by the biholomorphic mapping:

$$\chi_\mu : \begin{cases} z^* = \mu^{-1}z \\ w^* = \mu^{-2}w \end{cases}.$$

For an open neighborhood U of the origin and an open cone V_θ with its vertex at the origin and euclidean angle θ , $0 < \theta < \frac{\pi}{2}$, to the complex tangent hyperplane at the origin, we take θ_1 and a real number $\delta > 0$ such that

$$\chi_{\mu_1}(V_\theta \cap B(0; \delta)) \subset V_{\theta_1} \cap B(0; \delta_1).$$

Then, for each given curve $\eta : [0, 1] \rightarrow V_\theta \cap B(0; \delta)$, there is a continuous family of chain-segments

$$\gamma_1 : [0, 1] \times [0, 1] \rightarrow U_1 \cap M_{\mu_1}$$

where $\gamma_1(s, \cdot) : [0, 1] \rightarrow \chi_{\mu_1}(U \cap M)$ is a chain-segment of M_{μ_1} for each $s \in [0, 1]$ satisfying

$$\gamma_1(s, 0) = 0 \quad \text{and} \quad \gamma_1(s, 1) = \chi_{\mu_1}(\eta(s)) \quad \text{for all } s \in [0, 1].$$

Then, by Lemma 4.2, the desired family of chain-segments on M is given by

$$\gamma \equiv \chi_{\mu_1}^{-1} \circ \gamma_1 : [0, 1] \times [0, 1] \rightarrow U \cap M.$$

This completes the proof. \square

Theorem 4.8. *Let M be an analytic real hypersurface with nondegenerate Levi form and U be an open neighborhood of a point p of M . Then there are a number $\varepsilon > 0$ and a point $q \in U \cap M$ such that*

$$B(p; \varepsilon) \subset U$$

and, for each given curve $\eta : [0, 1] \rightarrow B(p; \varepsilon) \cap M$, there is a continuous family of chain-segments

$$\gamma : [0, 1] \times [0, 1] \rightarrow U \cap M$$

where $\gamma(s, \cdot) : [0, 1] \rightarrow U \cap M$ is a chain-segment of M for each $s \in [0, 1]$ satisfying

$$\gamma(s, 0) = q \quad \text{and} \quad \gamma(s, 1) = \eta(s) \quad \text{for all } s \in [0, 1].$$

Proof. We take a point q sufficiently near the point p such that there are an open cone V_θ and a number $\delta > 0$ in Lemma 4.7 satisfying

$$p \in V_\theta \cap B(q; \delta).$$

Then we take a number $\varepsilon > 0$ such that

$$B(p; \varepsilon) \subset V_\theta \cap B(q; \delta).$$

The desired result follows from Lemma 4.7. This completes the proof. \square

4.3. Piecewise chain curve. Let M be an analytic real hypersurface with non-degenerate Levi form. Let γ be a piecewise differentiable curve of $[0, 1]$ into M such that there are disjoint open intervals I_i , $i = 1, \dots, m$, satisfying

$$[0, 1] = \bigcup_{i=1}^m \overline{I_i}$$

and each fraction $\gamma(I_i)$, $i = 1, \dots, m$, is a chain-segment. Then γ shall be called a piecewise chain curve.

Lemma 4.9. *Let M be a connected analytic real hypersurface and Γ be a continuous curve on M connecting two points $p, q \in M$. Then, for a given number $\varepsilon > 0$, there is a piecewise chain curve $\gamma : [0, 1] \rightarrow M$ such that*

$$\begin{aligned} \gamma(0) &= p, \quad \gamma(1) = q \\ \gamma[0, 1] &\subset \bigcup_{x \in \Gamma} B(x; \varepsilon). \end{aligned}$$

Proof. Since the curve Γ is compact, there are finitely many points $x_i \in \Gamma$, $i = 1, \dots, l$, such that

$$\Gamma \subset \bigcup_{i=1}^l B(x_i; \varepsilon).$$

Suppose that x is a point on Γ and $x \in B(x_i; \varepsilon)$. Then, by Lemma 4.8, there is a number $\delta_x > 0$ such that every two points $y, z \in B(x; \delta_x)$ are connected by at most 2-pieced chain curve $\gamma \subset B(x_i; \varepsilon)$.

Note that the set $\{B(x; \delta_x) : x \in \Gamma\}$ is an open covering of Γ . Since Γ is compact, there is a finite subcover, say,

$$\Gamma \subset \bigcup_{j=1}^k B(y_j; \delta_{y_j}).$$

Then there is at most $2k$ -pieced chain curve $\gamma : [0, 1] \rightarrow M$ connecting the two point $p, q \in M$ such that

$$\gamma[0, 1] \subset \bigcup_{x \in \Gamma} B(x; \varepsilon).$$

This completes the proof. \square

Lemma 4.10. *Let M be a nondegenerate analytic real hypersurface. Suppose that there is a piecewise chain curve γ connecting two points $p, q \in M$. Then M is biholomorphic to a real hyperquadric at the point p if and only if M is biholomorphic to a real hyperquadric at the point q .*

Proof. Without loss of generality, we may assume that p and q are connected by a chain-segment $\gamma : [0, 1] \rightarrow M$. Then there is a chain Γ of M satisfying

$$\gamma[0, 1] \subset \Gamma.$$

For each point $x \in \Gamma$, there are an open neighborhood U_x of x and a biholomorphic mapping N_x such that

$$\begin{aligned} N_x(x) &= 0 \\ N_x(U_x \cap \Gamma) &\subset \{z = v = 0\}. \end{aligned}$$

Since the subset $\gamma[0, 1]$ is compact, there is a finite subcover, say,

$$\{U_{x_i} : i = 1, \dots, m\}.$$

Suppose that the normalization N_{x_i} transforms $M \cap U_{x_i}$ to the real hypersurface M'_{x_i} defined near the origin by

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}^i(z, \bar{z}, u).$$

Note that the functions $F_{st}^i(z, \bar{z}, u)$ are analytic of u on the set $N_{x_i}(U_{x_i} \cap \Gamma)$. Thus

$$F_{st}^i(z, \bar{z}, u) \equiv 0$$

whenever there is an open subset $U \subset U_{x_i}$ satisfying

$$F_{st}^i(z, \bar{z}, u) = 0 \quad \text{for } u \in N_{x_i}(U \cap \Gamma).$$

Thus the desired result follows. This completes the proof. \square

Theorem 4.11. *Let M be a connected nondegenerate analytic real hypersurface. Then M is not biholomorphic to a real hyperquadric at each point of M whenever there is a point p of M at which M is not biholomorphic to a real hyperquadric.*

Proof. The contrapositive may be stated as follows: M is locally biholomorphic to a real hyperquadric at each point of M whenever M is biholomorphic to a real hyperquadric at a point p of M . Suppose that there is a point p of M at which M is biholomorphic to a real hyperquadric. By Lemma 4.9, each point q of M is connected to p by a piecewise chain curve. Then, by Lemma 4.10, M is biholomorphic to a real hyperquadric at the point q as well. Since M is connected, this completes the proof. \square

Lemma 4.12. *Let M be an analytic real hypersurface and U be an open neighborhood of a point $p \in M$. Suppose that $U \cap M$ consists of umbilic points. Then $U \cap M$ is locally biholomorphic to a real hyperquadric.*

In the paper [Pa3], we have given the proof of Lemma 4.12.

Theorem 4.13. *Let M be a nondegenerate analytic real hypersurface and p be a point of M . Suppose that M is not biholomorphic to a real hyperquadric at the point p . Then there is a normalization ϕ near the point p such that $\phi(M)$ is defined by the equation, for $\dim M = 3$,*

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2, \max(s,t) \geq 4} F_{st}(z, \bar{z}, u)$$

where

$$F_{24}(z, \bar{z}, u) \neq 0,$$

and, for $\dim M \geq 5$,

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where

$$F_{22}(z, \bar{z}, u) \neq 0.$$

Proof. Suppose that the assertion is not true. Then M is umbilic on all points of all chains passing through the point p . Then, by Theorem 4.7, there is an open set U such that every point of $U \cap M$ is connected to p by a chain of M . Hence $U \cap M$ consists of umbilic points so that, by Lemma 4.12, $U \cap M$ is locally biholomorphic to a real hyperquadric. By Lemma 4.10, M is biholomorphic to a real hyperquadric at the point p as well. This is a contradiction. This completes the proof. \square

4.4. Global straightening of a chain. Let M be a nondegenerate analytic real hypersurface defined near the origin by the equation

$$v = \frac{1}{4\alpha} \ln \frac{1}{1 - 4\alpha \langle z, z \rangle} + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where α is a given real number and

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0.$$

By using the expansion

$$-\ln(1-x) = \sum_{m=1}^{\infty} \frac{x^m}{m},$$

the defining equation of M comes to

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}^*(z, \bar{z}, u)$$

where

$$\begin{aligned} \Delta F_{22}^*(z, \bar{z}, u) &= 4\alpha(n+1)\langle z, z \rangle \\ \Delta^2 F_{23}^*(z, \bar{z}, u) &= 0 \\ \Delta^3 F_{33}^*(z, \bar{z}, u) &= 32\alpha^2 n(n+1)(n+2). \end{aligned}$$

We may require the maximal analytic extension along the u -curve on the real hypersurface M in Moser-Vitushkin normal form.

Lemma 4.14. *Let M be a real hypersurface defined near the origin by*

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0.$$

Let \mathfrak{L} be the mapping

$$\mathfrak{L} : \begin{cases} z^* = \frac{z}{1-i\alpha w} \\ w^* = \frac{1}{2i\alpha} \ln \frac{1+i\alpha w}{1-i\alpha w} = \frac{1}{\alpha} \tan^{-1} \alpha w \end{cases}.$$

Then $\mathfrak{L}(M)$ is defined near the origin by the equation

$$v = \frac{1}{4\alpha} \ln \frac{1}{1 - 4\alpha \langle z, z \rangle} + \sum_{\min(s,t) \geq 2} F_{st}^*(z, \bar{z}, u)$$

where

$$\Delta F_{22}^* = \Delta^2 F_{23}^* = \Delta^3 F_{33}^* = 0.$$

Proof. Suppose that the real hypersurface M is in Chern-Moser normal form is defined by the equation

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where

$$(4.6) \quad \Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0.$$

Let \mathfrak{L} be a normalization of M to Moser-Vitushkin normal form leaving the u -curve invariant. We require the identity initial value on the normalization \mathfrak{L} so that \mathfrak{L} is necessarily of the form (cf. the proof of Theorem 0.4)

$$\mathfrak{L} : \begin{cases} z^* = \sqrt{q'(w)}U(w)z \\ w^* = q(w) \end{cases}$$

where

$$\langle U(u)z, U(u)z \rangle = \langle z, z \rangle \quad \text{and} \quad U(0) = id_{n \times n}.$$

Suppose that \mathfrak{L} transforms M to a real hypersurface M' defined by

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}^*(z, \bar{z}, u).$$

Then we obtain

$$(4.7) \quad \begin{aligned} F_{22}(z, \bar{z}, u) &= q'(u)F_{22}^*(U(u)z, \overline{U(u)z}, q(u)) \\ &\quad - i\langle z, z \rangle \langle z, U(u)^{-1}\{U'(u) + \frac{1}{2}q'(u)^{-1}q''(u)U(u)\}z \rangle \\ &\quad + i\langle z, z \rangle \langle U(u)^{-1}\{U'(u) + \frac{1}{2}q'(u)^{-1}q''(u)U(u)\}z, z \rangle \end{aligned}$$

and

$$(4.8) \quad F_{23}(z, \bar{z}, u) = q'(u)\sqrt{|q'(u)|}F_{23}^*(U(u)z, \overline{U(u)z}, q(u)).$$

The condition $\Delta^2 F_{23} = 0$ in (4.6) yields

$$\Delta^2 F_{23}^*(z, \bar{z}, u) = 0$$

so that the u -curve is a chain of M' . We require that M' is in Moser-Vitushkin normal form so that

$$\begin{aligned} \Delta F_{22}^*(z, \bar{z}, u) &= 4\alpha(n+1)\langle z, z \rangle \\ \Delta^2 F_{23}^*(z, \bar{z}, u) &= 0 \\ \Delta^3 F_{33}^*(z, \bar{z}, u) &= 32\alpha^2 n(n+1)(n+2). \end{aligned}$$

Then, in (4.7), we require the following condition

$$\begin{aligned} \Delta F_{22}(z, \bar{z}, u) &= 0 \\ \Delta F_{22}^*(z, \bar{z}, u) &= 4\alpha(n+1)\langle z, z \rangle \end{aligned}$$

so that

$$(4.9) \quad \begin{aligned} &4\alpha(n+1)q'(u)\langle z, z \rangle \\ &+ i(n+2)\{\langle U(u)^{-1}U'(u)z, z \rangle - \langle z, U(u)^{-1}U'(u)z \rangle\} \\ &+ i\langle z, z \rangle \{\text{Tr}(U(u)^{-1}U'(u)) - \overline{\text{Tr}(U(u)^{-1}U'(u))}\} \\ &= 0. \end{aligned}$$

From the condition $\langle U(u)z, U(u)z \rangle = \langle z, z \rangle$, we obtain

$$\begin{aligned} \langle U(u)^{-1}U'(u)z, z \rangle + \langle z, U(u)^{-1}U'(u)z \rangle &= 0 \\ \text{Tr}(U(u)^{-1}U'(u)) + \overline{\text{Tr}(U(u)^{-1}U'(u))} &= 0. \end{aligned}$$

The equality (4.9) comes to

$$2\alpha i(n+1)q'(u)id_{n \times n} = (n+2)U(u)^{-1}U'(u) + \text{Tr}(U(u)^{-1}U'(u))id_{n \times n},$$

which yields

$$\text{Tr}(U(u)^{-1}U'(u)) = \alpha niq'(u).$$

Hence we obtain

$$U'(u) = \alpha iq'(u)U(u).$$

Thus the function $U(u)$ is given by

$$U(u) = \exp \alpha iq(u).$$

Then the mapping \mathfrak{L} is necessarily of the form:

$$\mathfrak{L} : \begin{cases} z^* = \sqrt{q'(w)}z \exp \alpha iq(w) \\ w^* = q(w) \end{cases}.$$

Then we have

$$\begin{aligned} (4.10) \quad F_{33}(z, \bar{z}, u) &= q'(u) |q'(u)| F_{33}^*(z, \bar{z}, q(u)) \\ &\quad - 6\alpha q'(u)^2 \langle z, z \rangle F_{22}^*(z, \bar{z}, q(u)) \\ &\quad + \left\{ -\frac{q'''(u)}{3q'(u)} + \frac{1}{2} \left(\frac{q''(u)}{q'(u)} \right)^2 + 6\alpha^2 q'(u)^2 \right\} \langle z, z \rangle^3. \end{aligned}$$

We have the following identities:

$$\begin{aligned} \Delta^3 \langle z, z \rangle^3 &= 6n(n+1)(n+2) \\ \Delta^3 \{F_{33}^*(z, \bar{z}, q(u))\} &= \Delta^3 F_{33}^*(z, \bar{z}, q(u)) \\ \Delta^3 \{\langle z, z \rangle F_{22}^*(z, \bar{z}, q(u))\} &= 3(n+2)\Delta^2 F_{22}^*(z, \bar{z}, q(u)). \end{aligned}$$

Then, by requiring in (4.10) the following conditions

$$\Delta^3 F_{33}(z, \bar{z}, u) = 0$$

and

$$\begin{aligned} \Delta F_{22}^*(z, \bar{z}, u) &= 4\alpha(n+1)\langle z, z \rangle \\ \Delta^3 F_{33}^*(z, \bar{z}, u) &= 32\alpha^2 n(n+1)(n+2), \end{aligned}$$

we obtain

$$\frac{q'''(u)}{3q'(u)} - \frac{1}{2} \left(\frac{q''(u)}{q'(u)} \right)^2 + \frac{2\alpha^2}{3} \cdot q'(u)^2 = 0.$$

We easily check that the solution $q(u)$ with the initial value

$$q(0) = q''(0) = 0 \quad \text{and} \quad q'(0) = 1$$

is given by

$$q(w) = \frac{1}{2i\alpha} \ln \frac{1+i\alpha w}{1-i\alpha w} = \frac{1}{\alpha} \tan^{-1} \alpha w.$$

Then, with this $q(w)$, we easily check as well that

$$\sqrt{q'(w)} \exp \alpha i q(w) = \frac{1}{1 - i \alpha w}.$$

This completes the proof. \square

Lemma 4.15 (Ezhov). *Let M be an analytic real hypersurface in Moser-Vitushkin normal form and ϕ be a normalization of M to Moser-Vitushkin normal form such that ϕ leaves the u -curve invariant. Then the mapping ϕ is given by*

$$\phi : \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\}q'(w)} U z e^{\alpha i(q(w)-w)} \\ w^* = q(w) \end{cases}$$

where

$$\langle U z, U z \rangle = \text{sign}\{q'(0)\} \langle z, z \rangle$$

and the function $q(u)$ is a solution of the equation:

$$\frac{q'''}{3q'} - \frac{1}{2} \left(\frac{q''}{q'} \right)^2 + \frac{2\alpha^2}{3} (q'^2 - 1) = 0.$$

Proof. Suppose that the real hypersurface M is in Moser-Vitushkin normal form is defined by the equation

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}(z, \bar{z}, u)$$

where

$$\begin{aligned} \Delta F_{22}(z, \bar{z}, u) &= 4\alpha(n+1) \langle z, z \rangle \\ \Delta^2 F_{23}(z, \bar{z}, u) &= 0 \\ \Delta^3 F_{33}(z, \bar{z}, u) &= 32\alpha^2 n(n+1)(n+2). \end{aligned}$$

Let ϕ be a normalization of M to Moser-Vitushkin normal form leaving the u -curve invariant. Then the mapping ϕ is necessarily of the form(cf. the proof of Theorem 0.4)

$$\phi : \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\}q'(w)} U(w) z \\ w^* = q(w) \end{cases}$$

where

$$\langle U(u) z, U(u) z \rangle = \text{sign}\{q'(0)\} \langle z, z \rangle.$$

Suppose that ϕ transforms M to a real hypersurface M' defined by

$$v = \langle z, z \rangle + \sum_{\min(s,t) \geq 2} F_{st}^*(z, \bar{z}, u).$$

Then we obtain

$$\begin{aligned} F_{22}(z, \bar{z}, u) &= q'(u) F_{22}^*(U(u) z, \overline{U(u) z}, q(u)) \\ &\quad - i \langle z, z \rangle \langle z, U(u)^{-1} \{U'(u) + \frac{1}{2} q'(u)^{-1} q''(u) U(u)\} z \rangle \\ &\quad + i \langle z, z \rangle \langle U(u)^{-1} \{U'(u) + \frac{1}{2} q'(u)^{-1} q''(u) U(u)\} z, z \rangle \\ F_{23}(z, \bar{z}, u) &= q'(u) \sqrt{|q'(u)|} F_{23}^*(U(u) z, \overline{U(u) z}, q(u)). \end{aligned}$$

Then we easily see that

$$\Delta^2 F_{23}^*(z, \bar{z}, u) = 0.$$

We require the following condition

$$\begin{aligned} \Delta F_{22}(z, \bar{z}, u) &= \Delta F_{22}^*(z, \bar{z}, u) \\ &= 4\alpha(n+1)\langle z, z \rangle \end{aligned}$$

so that

$$\begin{aligned} &4\alpha(n+1)(q'(u) - 1)\langle z, z \rangle \\ &+ i(n+2)\{\langle U(u)^{-1}U'(u)z, z \rangle - \langle z, U(u)^{-1}U'(u)z \rangle\} \\ &+ i\langle z, z \rangle\{\text{Tr}(U(u)^{-1}U'(u)) - \overline{\text{Tr}(U(u)^{-1}U'(u))}\} \\ (4.11) \quad &= 0. \end{aligned}$$

From the equality

$$\langle U(u)z, U(u)z \rangle = \text{sign}\{q'(0)\}\langle z, z \rangle,$$

we have identities

$$\begin{aligned} \langle U(u)^{-1}U'(u)z, z \rangle + \langle z, U(u)^{-1}U'(u)z \rangle &= 0 \\ \text{Tr}(U(u)^{-1}U'(u)) + \overline{\text{Tr}(U(u)^{-1}U'(u))} &= 0. \end{aligned}$$

The equality (4.11) comes to

$$2\alpha i(n+1)(q'(u) - 1)id_{n \times n} = (n+2)U(u)^{-1}U'(u) + \text{Tr}(U(u)^{-1}U'(u))id_{n \times n},$$

which yields

$$\text{Tr}(U(u)^{-1}U'(u)) = \alpha ni(q'(u) - 1).$$

Hence we obtain

$$U'(u) = \alpha i(q'(u) - 1)U(u).$$

Thus the function $U(u)$ is given by

$$U(u) = U(0)e^{\alpha i(q(u) - u)}.$$

Then the mapping ϕ is necessarily of the form:

$$\phi : \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\}q'(w)}U(0)z \exp \alpha i(q(w) - w) \\ w^* = q(w) \end{cases}$$

where

$$\langle U(0)z, U(0)z \rangle = \text{sign}\{q'(0)\}\langle z, z \rangle.$$

Then we have

$$\begin{aligned} F_{33}(z, \bar{z}, u) &= q'(u) |q'(u)| F_{33}^*(U(0)z, \overline{U(0)z}, q(u)) \\ &\quad - 6\alpha q'(u)(q'(u) - 1)\langle z, z \rangle F_{22}^*(U(0)z, \overline{U(0)z}, q(u)) \\ (4.12) \quad &+ \left\{ -\frac{q'''(u)}{3q'(u)} + \frac{1}{2} \left(\frac{q''(u)}{q'(u)} \right)^2 + 3\alpha^2 (q'(u) - 1)^2 \right\} \langle z, z \rangle^3. \end{aligned}$$

We have the following identities:

$$\begin{aligned}\Delta^3 \langle z, z \rangle^3 &= 6n(n+1)(n+2) \\ \Delta^3 \left\{ F_{33}^*(U(0)z, \overline{U(0)z}, q(u)) \right\} &= \text{sign}\{q'(0)\} \Delta^3 F_{33}^*(z, \bar{z}, q(u)) \\ \Delta^3 \left\{ \langle z, z \rangle F_{22}^*(U(0)z, \overline{U(0)z}, q(u)) \right\} &= 3(n+2) \Delta^2 F_{22}^*(z, \bar{z}, q(u)).\end{aligned}$$

Then, requiring in (4.12) the following condition

$$\begin{aligned}\Delta^3 F_{33}(z, \bar{z}, u) &= \Delta^3 F_{33}^*(z, \bar{z}, u) \\ &= 32\alpha^2 n(n+1)(n+2),\end{aligned}$$

we obtain

$$\begin{aligned}\frac{q'''(u)}{3q'(u)} - \frac{1}{2} \left(\frac{q''(u)}{q'(u)} \right)^2 &= 6\alpha^2 (q'(u) - 1)^2 + \frac{16\alpha^2}{3} (q'(u)^2 - 1) \\ &\quad - 12\alpha^2 q'(u) (q'(u) - 1) \\ &= -\frac{2\alpha^2}{3} (q'(u)^2 - 1).\end{aligned}$$

This completes the proof. \square

Lemma 4.16 (Vitushkin). *Let $q(u)$ be an analytic solution of the equation*

$$(4.13) \quad \frac{q'''(u)}{3q'(u)} - \frac{1}{2} \left(\frac{q''(u)}{q'(u)} \right)^2 + \frac{2\alpha^2}{3} (q'(u)^2 - 1) = 0.$$

Then the function $q(u)$ is given by the relation

$$e^{2\alpha i q(u)} = e^{i\lambda} \frac{e^{2\alpha i u} + \kappa}{1 + \bar{\kappa} e^{2\alpha i u}}$$

where

$$\lambda \in \mathbb{R}, \quad \kappa \in \mathbb{C}, \quad |\kappa| \neq 1.$$

Further, the function $q(u)$ satisfies the relation

$$\left[\frac{q(u_2) - q(u_1)}{\pi\alpha^{-1}} \right] = \text{sign}\{q'(0)\} \left[\frac{u_2 - u_1}{\pi\alpha^{-1}} \right].$$

Proof. Let \mathfrak{L} be the mapping in Lemma 4.14. Then the normalization $\phi = (f^*, g^*)$ to Moser-Vitushkin normal form is given by the relation

$$\phi = \mathfrak{L} \circ \varphi \circ \mathfrak{L}^{-1}$$

where $\varphi = (f, g)$ is a normalization to Chern-Moser normal form. Explicitly, we obtain

$$\begin{cases} f^*(z, w) = \frac{f(z(1-i\tan\alpha w), \alpha^{-1}\tan\alpha w)}{1-i\alpha g(z(1-i\tan\alpha w), \alpha^{-1}\tan\alpha w)} \\ g^*(z, w) = \alpha^{-1}\tan^{-1}\alpha g(z(1-i\tan\alpha w), \alpha^{-1}\tan\alpha w) \end{cases}.$$

Here we take

$$\varphi : \begin{cases} z^* = \frac{Cz}{1-rw} \\ w^* = \frac{\rho w}{1-rw} \end{cases}$$

so that

$$\phi : \begin{cases} z^* = \sqrt{\text{sign}\{Q'(0)\} Q'(\exp 2\alpha i w)} Cz \\ w^* = \frac{1}{2\alpha i} \ln Q(\exp 2\alpha i w) \end{cases}$$

where

$$Q(w) = e^{i\lambda} \frac{w + \kappa}{1 + \bar{\kappa}w}$$

and

$$e^{i\lambda} = \frac{\alpha(1 + \rho) + ir}{\alpha(1 + \rho) - ir}, \quad \kappa = \frac{\alpha(1 - \rho) - ir}{\alpha(1 + \rho) + ir},$$

$$\rho = Q'(0) \neq 0, \quad \rho, r \in \mathbb{R}.$$

Then the solution $q(u)$ of the equation (4.13) is given by

$$q(u) = \frac{1}{2\alpha i} \ln Q(\exp 2\alpha i u)$$

so that

$$e^{2\alpha i q(u)} = e^{i\lambda} \frac{e^{2\alpha i u} + \kappa}{1 + \bar{\kappa}e^{2\alpha i u}}$$

where

$$\lambda \in \mathbb{R} \quad \text{and} \quad |\kappa| \neq 1.$$

Finally, note that the mapping

$$w^* = e^{i\lambda} \frac{w + \kappa}{1 + \bar{\kappa}w}$$

is an automorphism of the circle S^1 . Thus we obtain

$$\left[\frac{q(u_2) - q(u_1)}{\pi\alpha^{-1}} \right] = \text{sign}\{q'(0)\} \left[\frac{u_2 - u_1}{\pi\alpha^{-1}} \right].$$

This completes the proof. \square

Theorem 4.17 (Vitushkin). *Let M be a nondegenerate analytic real hypersurface and γ be a chain passing through the point p . Suppose that there are an open neighborhood U of p and a normalizing mapping ϕ of M to Moser-Vitushkin normal form such that ϕ translates the point p to the origin and*

$$\phi(\gamma \cap U) \subset \{z = v = 0\}.$$

Then the biholomorphic mapping ϕ of M is biholomorphically continued along γ such that γ is mapped by the mapping ϕ into the u -curve in Moser-Vitushkin normal form.

Proof. Let M' be a real hypersurface in Moser-Vitushkin normal form such that M' is maximally extended along the u -curve to the interval (u_-, u_+) , where

$$-\infty \leq u_- < 0 < u_+ \leq \infty.$$

Let ϕ be a normalizing mapping of M to M' such that γ is mapped by ϕ into the u -curve and the point p is mapped by ϕ to the origin. Then we claim that the mapping ϕ is biholomorphically continued along γ so that

$$(4.14) \quad \phi(\gamma) \subset (u_-, u_+).$$

Suppose that the assertion is not true. Then there is a chain-segment $\lambda : [0, 1] \rightarrow \gamma$ such that $\lambda(0) = p$ and ϕ is analytically continued along all subpath $\lambda[0, \tau]$, $\tau < 1$, but not the whole path $\lambda[0, 1]$. Let $q = \lambda(1)$. Since γ is a chain and q is an interior

point of γ , there are an open neighborhood V of the point q and a normalizing mapping h of M to Moser-Vitushkin normal form satisfying

$$\begin{aligned} h(q) &= 0 \\ h(V \cap \gamma) &\subset \{z = v = 0\}. \end{aligned}$$

We take a point x on $\lambda[0, 1]$ such that

$$x \in \lambda[0, 1] \cap U.$$

Then, by Lemma 4.15, there are an open neighborhood W of the point x and a biholomorphic mapping k satisfying

$$\phi = k \circ h \quad \text{on } W \cap V.$$

By Lemma 4.15 and Lemma 4.16, the mapping k is biholomorphically extended to an open neighborhood of the whole u -curve. Thus passing to an open subset of U containing $\lambda[0, 1] \cap U$, if necessary, the following mapping

$$k \circ h \quad \text{on } V$$

is an analytic continuation of ϕ over the point $\lambda(1)$. Then necessarily we have

$$k \circ h(\lambda[0, 1] \cap V) \subset (u_-, u_+).$$

This completes the proof. \square

4.5. Extension of a chain.

Lemma 4.18. *Let M be a nondegenerate analytic real hypersurface and $\gamma : [0, 1] \rightarrow M$ be a continuous curve. Then there exist a continuous family of real hypersurfaces M_τ , $\tau \in [0, 1]$, in normal form and a continuous family of biholomorphic mappings ϕ_τ , $\tau \in [0, 1]$, such that ϕ_τ translates the point $\gamma(\tau)$ to the origin and transforms the germ M at $\gamma(\tau)$ to the germ M_τ at the origin for each $\tau \in [0, 1]$ and the radius of convergence of the mapping ϕ_τ at the origin depends only on M and the point $\gamma(\tau)$.*

Proof. Without loss of generality, we may assume that the point $\gamma(0)$ is the origin and the real hypersurface M is defined near the origin by

$$v = F(z, \bar{z}, u), \quad F|_0 = F_z|_0 = F_{\bar{z}}|_0 = 0$$

and the curve $\gamma[0, 1]$ is given by some continuous functions $p(\tau)$ and $q(\tau)$ via the equation

$$\gamma : \begin{cases} z = p(\tau) \\ w = q(\tau) + iF(p(\tau), \bar{p}(\tau), q(\tau)) \end{cases} \quad \text{for } \tau \in [0, 1]$$

where

$$\overline{q(\tau)} = q(\tau).$$

Let φ_τ be a biholomorphic mapping defined by

$$\varphi_\tau : \begin{cases} z^* = z - p(\tau) \\ w^* = w - q(\tau) - 2i \sum_{\alpha=1}^n z^\alpha \left(\frac{\partial F}{\partial z^\alpha} \right) (p(\tau), \bar{p}(\tau), q(\tau)) \end{cases}.$$

Then we obtain the real hypersurfaces $\varphi_\tau(M)$, $\tau \in [0, 1]$, defined at the origin by the equation

$$v = F^\tau(z, \bar{z}, u), \quad F^\tau|_0 = F_z^\tau|_0 = F_{\bar{z}}^\tau|_0 = 0$$

where

$$\begin{aligned} F^\tau(z, \bar{z}, u) &= F(z + p(\tau), \bar{z} + \bar{p}(\tau), u + q^*(\tau)) \\ &\quad - F(p(\tau), \bar{p}(\tau), q(\tau)) \\ &\quad - \sum_{\alpha=1}^n z^\alpha \left(\frac{\partial F}{\partial z^\alpha} \right) (p(\tau), \bar{p}(\tau), q(\tau)) \\ &\quad - \sum_{\beta=1}^n \bar{z}^\beta \left(\frac{\partial F}{\partial \bar{z}^\beta} \right) (p(\tau), \bar{p}(\tau), q(\tau)) \end{aligned}$$

and

$$\begin{aligned} q^*(\tau) &= q(\tau) + i \sum_{\alpha=1}^n z^\alpha \left(\frac{\partial F}{\partial z^\alpha} \right) (p(\tau), \bar{p}(\tau), q(\tau)) \\ &\quad - i \sum_{\beta=1}^n \bar{z}^\beta \left(\frac{\partial F}{\partial \bar{z}^\beta} \right) (p(\tau), \bar{p}(\tau), q(\tau)). \end{aligned}$$

Let $\psi_\tau = (f^\tau, g^\tau)$ be a normalization of the germs $\varphi_\tau(M)$, $\tau \in [0, 1]$, with identity initial value such that the real hypersurface $\psi_\tau \circ \varphi_\tau(M)$ is defined by the equation

$$v = \langle z, z \rangle + \sum_{k=4}^{\infty} F_k^{*\tau}(z, \bar{z}, u).$$

By Theorem 0.6, the functions

$$\begin{cases} \tau \mapsto (f^\tau, g^\tau) \\ \tau \mapsto \sum_{k=4}^{\infty} F_k^{*\tau}(z, \bar{z}, u) \end{cases}$$

are continuous. Then the mappings $\phi_\tau = \psi_\tau \circ \varphi_\tau$ and the real hypersurfaces $M_\tau = \psi_\tau \circ \varphi_\tau(M)$ for each $\tau \in [0, 1]$ satisfy all the required conditions. This completes the proof. \square

Lemma 4.19. *Let M be a nondegenerate analytic real hypersurface and $\gamma : [0, 1] \rightarrow M$ be a curve such that $\gamma[0, \tau]$ is a chain-segment for each $\tau < 1$. Let U be an open set satisfying $\gamma[0, 1) \subset U$ and ϕ be a normalization of M on U to Moser-Vitushkin normal form. Suppose that there is a chain-segment λ on $\phi(M)$ in the u -curve satisfying*

$$\phi(\gamma[0, 1)) \subset \lambda.$$

Then

$$\sup_{0 \leq \tau < 1} \left\{ \frac{\left\| \left(\frac{\partial f}{\partial z} \right|_{\phi \circ \gamma(\tau)} \right) \right\|}{\left(\left| \frac{\partial g}{\partial w} \right|_{\phi \circ \gamma(\tau)} \right)^{\frac{1}{2}}}, \quad \frac{\left\| \left(\frac{\partial f}{\partial \bar{z}} \right|_{\phi \circ \gamma(\tau)} \right)^{-1} \right\|}{\left(\left| \frac{\partial g}{\partial w} \right|_{\phi \circ \gamma(\tau)} \right)^{-\frac{1}{2}}} \right\} < \infty$$

where

$$\phi^{-1} = (f, g).$$

Proof. Suppose that the real hypersurface M is defined on an open neighborhood U of the origin by the equation

$$v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0$$

and the curve $\gamma[0, 1] \subset M \cap U$ is passing through the origin. Then there is a biholomorphic mapping

$$\phi : \begin{cases} z = z^* + D(z^*, w^*) \\ w = w^* + g(z^*, w^*) \end{cases}$$

where

$$D_z(0, u) = 0, \quad \Re g(0, u) = 0$$

such that the mapping ϕ straightens $\gamma[0, 1]$ into the u -curve and transforms M to a real hypersurface $\phi(M)$ defined by

$$v = F_{11}^*(z, \bar{z}, u) + \sum_{s, t \geq 2} F_{st}^*(z, \bar{z}, u)$$

where

$$(\text{tr})^2 F_{23} = 0.$$

Note that $F_{11}^*(z, \bar{z}, u_\tau)$ is the Levi form of M at the point $\gamma(\tau) \in U \cap M$ for $\tau \in [0, 1]$, where

$$(0, u_\tau) = \phi \circ \gamma(\tau).$$

Since λ is a chain-segment, $F_{11}^*(z, \bar{z}, u_\tau)$ may be finite for all $\tau \in [0, 1]$. Thus we can take a matrix $E_1(u)$ and a real number $c > 0$ such that

$$F_{11}^*(z, \bar{z}, u) = \langle E_1(u) z, E_1(u) z \rangle$$

and

$$\sup_{\tau \in [0, 1]} \left\{ \|E_1(u_\tau)\|, \quad \|E_1(u_\tau)^{-1}\| \right\} \leq c < \infty.$$

We shall show

$$\sup_{0 \leq \tau < 1} \left\{ \|E(u_\tau)\|, \quad \|E(u_\tau)^{-1}\| \right\} < \infty$$

where $\phi^{-1} = (f, g)$ and

$$E(u) = \frac{\left(\frac{\partial f}{\partial z} \Big|_{z=v=0} \right)}{\sqrt{\left| \frac{\partial g}{\partial w} \Big|_{z=v=0} \right|}}.$$

Here the function $E(u)$ satisfies the following ordinary differential equation(cf. [Pa2])

$$\begin{aligned} & F_{11}^* \left(E(u)^{-1} E'(u) z, \bar{z}, u \right) \\ &= -\frac{2i}{n+1} \cdot \text{tr} F_{22}^*(z, \bar{z}, u) + \frac{1}{2} \left(\frac{\partial F_{11}^*}{\partial u} \right) (z, \bar{z}, u) \\ & \quad + \frac{i}{(n+1)(n+2)} \cdot (\text{tr})^2 F_{22}^* \times F_{11}^*(z, \bar{z}, u). \end{aligned}$$

We easily see that there is a real number $e > 0$ such that

$$\sup_{0 \leq \tau < 1} \|E(u_\tau)^{-1} E'(u_\tau)\| \leq e < \infty.$$

Notice that

$$E(u)^{-1} E'(u) = - (E(u)^{-1})' E(u).$$

Because λ is a chain-segment, we have

$$\int_{\phi \circ \gamma[0,1]} du \leq \int_{\lambda} du < \infty.$$

Hence we obtain the following estimates

$$\begin{aligned} \|E(u_\tau)\| &\leq \|E(u_0)\| \exp \int_{\phi \circ \gamma[0,1]} e du < \infty \\ \|E(u_\tau)^{-1}\| &\leq \|E(u_0)^{-1}\| \exp \int_{\phi \circ \gamma[0,1]} e du < \infty \end{aligned}$$

where

$$(0, u_0) = \phi \circ \gamma(0).$$

So the condition $F_{11}^*(z, \bar{z}, u) = \langle E(u)z, E(u)z \rangle$ determines the matrix $E(u)$ such that

$$E(u) = U(u) E_1(u)$$

where $U(u)$ is any matrix satisfying

$$\langle U(u)z, U(u)z \rangle = \langle z, z \rangle.$$

Hence we have the following relation

$$\begin{aligned} c^{-1} \|E(u_\tau)\| &\leq \|U(u_\tau)\| \leq c \|E(u_\tau)\| \\ c^{-1} \|E(u_\tau)^{-1}\| &\leq \|U(u_\tau)^{-1}\| \leq c \|E(u_\tau)^{-1}\| \end{aligned}$$

for all $\tau \in [0, 1]$. Therefore, we also have showed

$$\sup_{\tau \in [0,1]} \left\{ \|U(u_\tau)\|, \|U(u_\tau)^{-1}\| \right\} < \infty.$$

This completes the proof. \square

Lemma 4.20. *Let M be an analytic real hypersurface in normal form defined by*

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u)$$

where

$$F^*(z, \bar{z}, u) = \sum_{k=4}^{\infty} F_k^*(z, \bar{z}, u).$$

Suppose that M is not a real hyperquadric. Then there is an integer $l \geq 4$ such that

$$\begin{aligned} F_k^*(z, \bar{z}, u) &= 0 \quad \text{for all } k \leq l-1 \\ F_l^*(z, \bar{z}, u) &\neq 0 \end{aligned}$$

for any value of U, a, ρ, r .

In the paper [Pa3], we have given the proof of Lemma 4.20.

Theorem 4.21. *Let M, M' be nonspherical analytic real hypersurfaces and $\gamma : [0, 1] \rightarrow M$ be a curve such that $\gamma[0, \tau]$ is a chain-segment for each $\tau < 1$. Let U be an open neighborhood of $\gamma[0, 1)$ and ϕ be a biholomorphic mapping on U such that ϕ transforms M to a real hypersurface M' satisfying*

$$\phi(U \cap M) \subset M'$$

and there is a chain-segment $\lambda : [0, 1] \rightarrow M'$ satisfying

$$\phi(\gamma[0, 1]) \subset \lambda.$$

Suppose that there is a real number $c \geq 1$ such that

$$(4.15) \quad \sup_{0 \leq \tau \leq 1} \sup_{(U_\tau, 0, \rho_\tau, r_\tau) \in H_{\lambda(\tau)}(M')} \|U_\tau\| \leq c < \infty$$

where $H_{\lambda(\tau)}(M')$ is the local automorphism group of M' at the point $\lambda(\tau)$ in a normal coordinate with the chain-segment λ on the u -curve. Then there exists a chain Γ on M satisfying

$$\gamma[0, 1] \subset \Gamma,$$

i.e., $\gamma[0, 1]$ is a chain-segment.

Proof. Without loss of generality, we may assume that the real hypersurface M' is in Moser-Vitushkin normal form with the chain-segment λ on the u -curve so that M' is defined by the equation

$$v = \frac{1}{4\alpha} \ln \frac{1}{1 - 4\alpha \langle z, z \rangle} + \sum_{k=4}^{\infty} G_k(z, \bar{z}, u).$$

Here we assume $\alpha \neq 0$ and later we shall take a sufficiently small value for α .

There exists a continuous function $\tau \mapsto u_\tau$ for $\tau \in [0, 1]$ such that

$$(0, u_\tau) = \phi(\gamma(\tau)) \subset \lambda \quad \text{for } \tau \in [0, 1].$$

Since the chain-segment λ is compact, there is a real number u_1 such that

$$(0, u_1) = \lim_{\tau \rightarrow 1} \phi(\gamma(\tau)) \in \lambda.$$

Then we obtain a continuous family of analytic real hypersurfaces M'_τ , $\tau \in [0, 1]$, defined near the origin by

$$v = \frac{1}{4\alpha} \ln \frac{1}{1 - 4\alpha \langle z, z \rangle} + G^\tau(z, \bar{z}, u)$$

where, for $\tau \in [0, 1]$,

$$\begin{aligned} G^\tau(z, \bar{z}, u) &= \sum_{k=4}^{\infty} G_k(z, \bar{z}, u + u_\tau) \\ &= \sum_{k=4}^{\infty} G_k^\tau(z, \bar{z}, u). \end{aligned}$$

By Lemma 4.18, we obtain a continuous family of analytic real hypersurfaces M_τ , $\tau \in [0, 1]$, in normal form and a continuous family of biholomorphic mappings φ_τ for the real hypersurface M and the curve $\gamma : [0, 1] \rightarrow M$. Then, for each $\tau \in [0, 1]$, there exist an open neighborhood V_τ of the origin and a chain γ_τ on M_τ passing through the origin such that

$$\varphi_\tau^{-1}(V_\tau \cap \gamma_\tau) \subset \gamma[0, 1].$$

Suppose that M_τ , $\tau \in [0, 1]$, is defined in normal form by

$$v = \langle z, z \rangle + \sum_{k=4}^{\infty} F_k^\tau(z, \bar{z}, u).$$

By Lemma 4.20, there is a well-defined integer m_τ , $\tau \in [0, 1]$, such that

$$\begin{cases} F_k^\tau(z, \bar{z}, u) = 0 & \text{for } k \leq m_\tau - 1 \\ F_{m_\tau}^\tau(z, \bar{z}, u) \neq 0 \end{cases}$$

because M_τ is nonspherical.

Let ϕ_τ be a normalization of M_τ for each $\tau \in [0, 1)$ to Moser-Vitushkin normal form such that the initial value σ of the mapping ϕ_τ is given by

$$\sigma = (id_{n \times n}, a_\tau, 1, 0)$$

where a_τ is determined by the condition

$$\phi_\tau(\gamma_\tau \cap M_\tau) \subset \{z = v = 0\}$$

Suppose that $\phi_\tau(M_\tau)$, $\tau \in (0, 1)$, is defined near the origin by the equation

$$v = \frac{1}{4\alpha} \ln \frac{1}{1 - 4\alpha \langle z, z \rangle} + F^{*\tau}(z, \bar{z}, u; a_\tau)$$

where

$$F^{*\tau}(z, \bar{z}, u; a_\tau) = \sum_{k=4}^{\infty} F_k^{*\tau}(z, \bar{z}, u; a_\tau).$$

Notice that the function $\tau \mapsto a_\tau$ is continuous on $[0, 1)$ and, by Theorem 0.6, the function

$$\tau \mapsto F^{*\tau}(z, \bar{z}, u; a)$$

is continuous on $[0, 1)$ for a fixed $a \in \mathbb{C}^n$.

Note that the two real hypersurfaces $\phi_\tau(M_\tau)$ and M'_τ are in Moser-Vitushkin normal form and biholomorphic to each other at the origin for all $\tau \in [0, 1)$ by a biholomorphic mapping leaving the u -curve invariant. Thus there is a mapping

$$\psi_\tau : \begin{cases} z^* = \sqrt{q'_\tau(w)} U_\tau z \exp \alpha i(q_\tau(w) - w) \\ w^* = q_\tau(w) \end{cases}$$

such that

$$\phi_\tau(M_\tau) = \psi_\tau(M'_\tau) \quad \text{for all } \tau \in [0, 1).$$

Then the function $q_\tau(u)$ is a solution of the ordinary differential equation

$$\frac{q'''}{3q'} - \frac{1}{2} \left(\frac{q''}{q'} \right)^2 + \frac{2\alpha^2}{3} (q'^2 - 1) = 0$$

with the initial conditions

$$\Re q(0) = 0, \quad \Re q'(0) = \rho_\tau \in \mathbb{R}^+, \quad \Re q''(0) = 2\rho_\tau r_\tau \in \mathbb{R}.$$

Suppose that $\psi_\tau(M'_\tau)$, $\tau \in [0, 1)$, is defined by the equation

$$v = \frac{1}{4\alpha} \ln \frac{1}{1 - 4\alpha \langle z, z \rangle} + G^{*\tau}(z, \bar{z}, u)$$

where

$$G^{*\tau}(z, \bar{z}, u) = \sum_{k=4}^{\infty} G_k^{*\tau}(z, \bar{z}, u; U_\tau, \rho_\tau, r_\tau).$$

Since $\phi_\tau(M_\tau) = \psi_\tau(M'_\tau)$ for $\tau \in [0, 1)$, we obtain

$$(4.16) \quad F_k^{*\tau}(z, \bar{z}, u; a_\tau) = G_k^{*\tau}(z, \bar{z}, u; U_\tau, \rho_\tau, r_\tau) \quad \text{for } k \geq 4.$$

We take a sequence $\tau_j, j \in \mathbb{N}$, such that

$$\tau_j \in [0, 1) \quad \text{and} \quad \tau_j \nearrow 1.$$

Then there exist a matrix U_{τ_j} and a function $q_{\tau_j}(u)$ such that

$$\psi_{\tau_j} : \begin{cases} z^* = \sqrt{q'_{\tau_j}(w)} U_{\tau_j} z \exp \alpha i (q_{\tau_j}(w) - w) \\ w^* = q_{\tau_j}(w) \end{cases}.$$

Lemma 4.19 and the condition (4.15) allow us to assume

$$\sup_j \|U_{\tau_j}\| < \infty$$

so that, passing to a subsequence, if necessary, there exist a matrix U satisfying

$$U = \lim_{j \rightarrow \infty} U_{\tau_j}.$$

By Lemma 4.16, all the functions $q_{\tau_j}(u)$ satisfy the following estimate

$$\begin{aligned} |q_{\tau_j}(u)| &= |q_{\tau_j}(u) - q_{\tau_j}(0)| \\ &\leq \pi |\alpha|^{-1} \left\{ \left[\frac{|q_{\tau_j}(u) - q_{\tau_j}(0)|}{\pi |\alpha|^{-1}} \right] + 1 \right\} \\ &\leq \pi |\alpha|^{-1} \left\{ \left[\frac{|u|}{\pi |\alpha|^{-1}} \right] + 2 \right\} \\ &\leq |u| + 2\pi |\alpha|^{-1}. \end{aligned}$$

Because λ is a chain-segment on M' , the functions $q_{\tau_j}(u)$ are bounded in the range which we have interested in. Further, notice that

$$q_{\tau_j}(\pi \alpha^{-1}) = \pm \pi \alpha^{-1} \quad \text{for all } j.$$

Then, passing to a subsequence, if necessary, Montel theorem and Hurwitz theorem allow us to have a function $q(u)$ such that

$$q(u) = \lim_{j \rightarrow \infty} q_{\tau_j}(u)$$

and

$$q(0) = 0 \quad \text{and} \quad q'(0) \neq 0.$$

Hence, passing to a subsequence, if necessary, there is a real number $e > 0$ such that

$$\sup_j \left\{ \|U_{\tau_j}\|, \|U_{\tau_j}^{-1}\|, |\rho_{\tau_j}|, |\rho_{\tau_j}^{-1}|, |r_{\tau_j}| \right\} \leq e < \infty.$$

By the definition of the integer m_1 , we have

$$\begin{cases} \lim_{\tau \rightarrow 1} F_k^\tau(z, \bar{z}, u) = 0, & k = 4, \dots, m_1 - 1, \\ \lim_{\tau \rightarrow 1} F_{m_1}^\tau(z, \bar{z}, u) \neq 0. \end{cases}$$

Then the function $F_{m_1+1}^{*\tau}(z, \bar{z}, u; a_\tau)$ may be decomposed to three parts as follows

(4.17)

$$F_{m_1+1}^{*\tau}(z, \bar{z}, u; a_\tau) = F_{m_1+1}^\tau(z, \bar{z}, u) + H_{m_1+1}^\tau(z, \bar{z}, u; a_\tau) + L_{m_1+1}^\tau(z, \bar{z}, u; a_\tau)$$

where

- (1) the function $H_{m_1+1}^\tau(z, \bar{z}, u; a)$ is determined by the function $F_{m_1}^\tau(z, \bar{z}, u)$,

- (2) the function $L_{m_1+1}^\tau(z, \bar{z}, u; a)$ is determined by the functions $F_k^\tau(z, \bar{z}, u)$, $k \leq m_1 - 1$,
 (3) the function $H_{m_1+1}^\tau(z, \bar{z}, u; a)$ is linear with respect to a and the mapping

$$a \longmapsto \lim_{\alpha \rightarrow 0} \lim_{\tau \rightarrow 1} H_{m_1+1}^\tau(z, \bar{z}, u; a)$$

is injective(cf. Lemma 2.8), where α is the parameter of Moser-Vitushkin normal form,

- (4) the function $L_{m_1+1}^\tau(z, \bar{z}, u; a)$ depends polynomially on the parameter a and

$$\lim_{\alpha \rightarrow 0} \lim_{\tau \rightarrow 1} L_{m_1+1}^\tau(z, \bar{z}, u; a) = 0 \quad \text{for any fixed } a \in \mathbb{C}^n.$$

Notice that there is a real number $\varepsilon_1 > 0$ such that the mapping

$$a \longmapsto H_{m_1+1}^\tau(z, \bar{z}, u; a)$$

is injective for all $|\alpha| \leq \varepsilon_1$ and all $\tau \geq 1 - \varepsilon_1$. We take a value for the parameter α such that $0 < |\alpha| \leq \varepsilon_1$.

Then the equalities (4.16) and (4.17) yields

$$\begin{aligned} & H_{m_1+1}^{\tau_j}(z, \bar{z}, u; a_{\tau_j}) + L_{m_1+1}^{\tau_j}(z, \bar{z}, u; a_{\tau_j}) \\ &= G_{m_1+1}^{*\tau_j}(z, \bar{z}, u; U_{\tau_j}, \rho_{\tau_j}, r_{\tau_j}) - F_{m_1+1}^{\tau_j}(z, \bar{z}, u). \end{aligned}$$

By taking smaller $\varepsilon_1 > 0$, if necessary, the injectivity of the mapping $a \longmapsto H_{m_1+1}^\tau(z, \bar{z}, u; a)$ allows to take an estimate of a_{τ_j} such that there is a real number $c_1 > 0$ and

$$|a_{\tau_j}| \leq c_1 < \infty \quad \text{for all } \tau_j, j \in \mathbb{N}.$$

Notice that the function $\tau \rightarrow a_\tau$ is continuous. Thus there exists a real number $c > 0$ such that

$$|a_\tau| \leq c < \infty \quad \text{for all } \tau \in [0, 1].$$

Therefore, there exists a sufficiently small real number $\delta > 0$ independent of $\tau \in [0, 1]$ such that the real hypersurface M_τ and the chain $\gamma_\tau \subset M_\tau$ extend to

$$\begin{aligned} M_\tau &= \phi_\tau^{-1}(\psi_\tau(M'_\tau) \cap B(0; \delta)) \\ \gamma_\tau &= \phi_\tau^{-1}(\{z = v = 0\} \cap B(0; \delta)) \end{aligned}$$

and the mappings φ_τ^{-1} extends biholomorphically on $B(0; \delta)$.

Then there is a sufficiently small real numbers $\varepsilon > 0$ such that

$$\gamma(1) \in \varphi_{1-\varepsilon}^{-1}(B(0; \delta))$$

so that the following curve Γ defined by

$$\Gamma = \gamma[0, 1) \cup \varphi_{1-\varepsilon}^{-1}(\gamma_{1-\varepsilon})$$

is a chain on M such that

$$\gamma[0, 1] \subset \Gamma.$$

This completes the proof. \square

Note that the condition (4.15) is trivially satisfied if the Levi form on the real hypersurface M is definite.

Theorem 4.22. *Let M be a strongly pseudoconvex analytic real hypersurface and γ be a chain on M . Let $\Gamma : (0, 1) \rightarrow M$ be the maximally extended connected open analytic curve on M containing the chain γ and Γ_0 be a maximal subarc of Γ such that Γ_0 contains the chain γ and Γ_0 is transversal to the complex tangent hyperplane of M at each point of Γ_0 . Then Γ_0 is a chain, i.e., for each point $p \in \Gamma_0$, there exist an open neighborhood U of the point p and a biholomorphic mapping ϕ on U such that*

$$\phi(U \cap \Gamma) \subset \{z = v = 0\}$$

and the mapping ϕ translates the point p to the origin and transforms the germ M at the point p to normal form.

Proof. Let M' be a real hypersurface in Moser-Vitushkin normal form such that M' is maximally extended along the u -curve to the interval (u_-, u_+) , where

$$-\infty \leq u_- < 0 < u_+ \leq \infty.$$

Suppose that there is a normalizing mapping ϕ of M to M' such that γ is mapped by ϕ into the u -curve. Then, by Theorem 4.17, the mapping ϕ is biholomorphically continued along γ so that

$$\phi(\gamma) \subset (u_-, u_+).$$

By Theorem 4.21, the chain γ can be extended on M , say, to a chain $\Gamma \subset M$, whenever an end limit of γ exists on M and the corresponding end limit of $\phi(\gamma)$ is an interior point of (u_-, u_+) . By Theorem 4.17, the mapping ϕ is biholomorphically continued along Γ so that

$$\phi(\Gamma) \subset (u_-, u_+).$$

Hence there exists a unique chain $\Gamma : (0, 1) \rightarrow M$ maximally extended from the chain γ such that

$$\lim_{\tau \rightarrow 0} \Gamma(\tau) \notin M \quad \text{or} \quad \lim_{\tau \rightarrow 0} \phi(\Gamma(\tau)) \in \{u_-, u_+\}$$

and

$$\lim_{\tau \rightarrow 1} \Gamma(\tau) \notin M \quad \text{or} \quad \lim_{\tau \rightarrow 1} \phi(\Gamma(\tau)) \in \{u_-, u_+\}.$$

Suppose that

$$\lim_{\tau \rightarrow 0} \Gamma(\tau) \in M \quad \text{and} \quad \lim_{\tau \rightarrow 0} \phi(\Gamma(\tau)) \in \{u_-, u_+\}.$$

We claim that the analytic curve $\Gamma : (0, 1) \rightarrow M$ is not analytically continued over the limit

$$q = \lim_{\tau \rightarrow 0} \Gamma(\tau) \in M$$

transversely to the complex tangent hyperplane of M at the point $q \in M$. Otherwise, there exist an open neighborhood U of the point q and an analytic curve $\lambda : (-1, 1) \rightarrow M$ such that

$$U \cap \lambda(0, 1) \subset \Gamma(0, 1) \quad \text{and} \quad \lambda(0) = q$$

and λ is transversal to the complex tangent hyperplanes of M at each point of λ . Then there exist a sufficiently small real number $\varepsilon > 0$ such that the analytic curve

$$\lambda(-\varepsilon, \varepsilon) \cup \Gamma(0, 1)$$

is an chain as well. Then M' is analytically extended along the u -curve over the point u_- or u_+ . This is a contradiction to the definition of the point u_- and u_+ .

Therefore, the chain Γ is the maximally extended connected open analytic curve containing the chain γ which is transversal to the complex tangent hyperplanes of M at each point of Γ . This completes the proof. \square

5. ANALYTIC CONTINUATION OF A BIHOLOMORPHIC MAPPING

5.1. On a spherical real hypersurface.

Theorem 5.1 (Pinchuk, Chern-Ji). *Let M be a spherical analytic real hypersurfaces with definite Levi form in a complex manifold, U be a connected neighborhood of a point $p \in M$, and ϕ be a biholomorphic mapping such that $\phi(U \cap M) \subset S^{2n+1}$. Then the mapping ϕ continues holomorphically along any path in M as a locally biholomorphic mapping.*

Proof. Suppose that the assertion is not true. Then there would exists a path $\gamma[0, 1]$ such that a biholomorphic mapping ϕ at the point $p = \gamma(0)$ can be biholomorphically continued along all subpath $\gamma[0, \tau]$ with $\tau < 1$, but not along the whole path. We set $q = \gamma(1)$. Since M is spherical, every point of M is umbilic. By Lemma 4.12, there is an open subset U_q of the point q and a biholomorphic mapping h_q on U_q such that

$$h_q(U_q \cap M) \subset S^{2n+1}$$

and we can take τ satisfying $\gamma(t) \in U_q \cap M$ for all $t \in [\tau, 1]$. Then there are an open neighborhood U of the point $\gamma(\tau)$ and a unique automorphisms φ of S^{2n+1} such that

$$\phi = \varphi \circ h_q \quad \text{on } U \cap U_q.$$

By a classical theorem of Poincaré, φ is biholomorphic on an open neighborhood of S^{2n+1} . Thus passing to an open subset of U_q containing $\gamma[0, 1] \cap U_q$, if necessary, $\varphi \circ h_q$ is an analytic continuation of ϕ on U_q . This is a contradiction. This completes the proof. \square

Theorem 5.2 (Pinchuk). *Let D be a bounded strongly pseudoconvex domain in \mathbb{C}^{n+1} with simply connected real-analytic boundary ∂D . Suppose that ∂D is a spherical analytic real hypersurface. Then there is a biholomorphic mapping ϕ of D onto B^{n+1} .*

Proof. By Lemma 4.12, ∂D is locally biholomorphic to S^{2n+1} . We take a point $p \in \partial D$ and an open neighborhood U of p such that there is a biholomorphic mapping ϕ on U satisfying $\phi(U \cap \partial D) \subset S^{2n+1}$. Then, by Theorem 5.1, the mapping ϕ extends along any path on ∂D as a local biholomorphic mapping. Since ∂D is simply connected, the monodromy theorem yields a unique biholomorphic extension ϕ , by keeping the same notation, on an open neighborhood of ∂D .

Note that $\phi : \partial D \rightarrow S^{2n+1}$ is an open mapping because ϕ is biholomorphic on an open neighborhood of ∂D . Since ∂D is compact, the mapping $\phi : \partial D \rightarrow S^{2n+1}$ is a covering map. Further, since S^{2n+1} is simply connected, the mapping $\phi : \partial D \rightarrow S^{2n+1}$ is a simple covering map so that there exists a biholomorphic inverse $\phi^{-1} : S^{2n+1} \rightarrow \partial D$. By Hartogs extension theorem, the mappings ϕ, ϕ^{-1} extend to open neighborhoods respectively of \overline{D} and $\overline{B^{n+1}}$. Thus the mapping ϕ induces a biholomorphic mapping of D onto B^{n+1} . This completes the proof. \square

Theorem 5.3. *Let D be simply connected open set in a complex manifold with compact simply connected real-analytic boundary ∂D and compact closure \overline{D} . Suppose that ∂D is a spherical analytic real hypersurface. Then there is a biholomorphic mapping ϕ of D onto B^{n+1} .*

Proof. By the same argument, there is a biholomorphic mapping ϕ on an open neighborhood of the boundary ∂D such that $\phi : \partial D \rightarrow S^{2n+1}$ is a simple covering map. Thus there exists a biholomorphic inverse $\phi^{-1} : S^{2n+1} \rightarrow \partial D$. By Hartogs extension theorem, the mapping ϕ^{-1} extends to the open ball B^{n+1} as a local biholomorphic mapping. Since $\overline{B^{n+1}}$ and \overline{D} are compact, the mapping $\phi^{-1} : B^{n+1} \rightarrow D$ is a covering mapping. Since D is simply connected, $\phi^{-1} : B^{n+1} \rightarrow D$ is a simple covering map. Thus the mapping ϕ induces a biholomorphic mapping of D onto B^{n+1} . This completes the proof. \square

Let Q be a real hyperquadric in \mathbb{CP}^{n+1} which is defined in a homogeneous coordinate

$$(\eta, \zeta^1, \dots, \zeta^n, \xi) \in \mathbb{C}^{n+2}$$

by the equation

$$\frac{1}{2i} (\xi \overline{\eta} - \eta \overline{\xi}) = \langle \zeta, \zeta \rangle$$

where

$$\langle \zeta, \zeta \rangle \equiv \zeta^1 \overline{\zeta^1} + \dots + \zeta^n \overline{\zeta^n} - \dots - \zeta^n \overline{\zeta^n}.$$

Then the real hyperquadric Q is given in the inhomogeneous coordinate chart $\simeq \mathbb{C}^{n+1}$ by the equation

$$\frac{1}{2i} (w - \overline{w}) = \langle z, z \rangle$$

where

$$z = \left(\frac{\zeta^1}{\eta}, \dots, \frac{\zeta^n}{\eta} \right), \quad w = \frac{\xi}{\eta}.$$

Lemma 5.4 (Chern-Moser). *Let Q be a real hyperquadric in \mathbb{CP}^{n+1} and U be an open neighborhood of a point $p \in Q$. Suppose that there is a biholomorphic mapping ϕ on U such that $\phi(U \cap Q) \subset Q$. Then the mapping ϕ extends to be an automorphism of Q which is biholomorphic on an open neighborhood of Q .*

Proof. By composing a linear mapping of \mathbb{CP}^{n+1} to ϕ , if necessarily, we may assume that ϕ has a fixed point $q \in Q$. Further, passing to an inhomogeneous coordinate chart, ϕ is a local automorphism of the real hyperquadric $v = \langle z, z \rangle$ in \mathbb{C}^{n+1} . By Theorem 0.4, the mapping ϕ is necessarily to be a fractional linear mapping as follows:

$$\begin{aligned} z^* &= \frac{C(z - aw)}{1 + 2i\langle z, a \rangle - w(r + i\langle a, a \rangle)} \\ w^* &= \frac{\rho w}{1 + 2i\langle z, a \rangle - w(r + i\langle a, a \rangle)}. \end{aligned}$$

Thus the mapping ϕ extends to be a linear mapping in \mathbb{CP}^{n+1} . This completes the proof. \square

Theorem 5.5. *Let M be a spherical analytic real hypersurface with nondefinite Levi form in a complex manifold, U be a connected neighborhood of a point $p \in M$, and ϕ be a biholomorphic mapping on U such that*

$$\phi(U \cap M) \subset Q$$

where Q is a real hyperquadric in \mathbb{CP}^{n+1} . Then the mapping ϕ continues holomorphically along any path on M as a locally biholomorphic mapping.

Proof. Suppose that the assertion is not true. Then there would exist a path $\gamma[0, 1]$ with $p = \gamma(0)$ such that a biholomorphic mapping ϕ on the neighborhood U of p can be biholomorphically continued along all subpath $\gamma[0, \tau]$ with $\tau < 1$, but not along the whole path. We set $q = \gamma(1)$. Since M is spherical, by Lemma 4.12, there is an open subset U_q of the point q and a biholomorphic mapping h_q on U_q such that

$$h_q(U_q \cap M) \subset Q.$$

We take τ satisfying $\gamma(t) \in U_q \cap M$ for all $t \in [\tau, 1]$. Then there are an open neighborhood U of the point $\gamma(\tau)$ and a unique automorphism φ of Q such that

$$\phi = \varphi \circ h_q \quad \text{on } U \cap U_q.$$

By Lemma 5.4, φ is biholomorphic on an open neighborhood of Q . Then passing to an open subset of U_q containing $\gamma[0, 1] \cap U_q$, if necessary, $\varphi \circ h_q$ is an analytic continuation of ϕ on U_q . This is a contradiction. This completes the proof. \square

5.2. On a nonspherical real hypersurface.

Lemma 5.6. *Let M, M' be nonspherical analytic real hypersurfaces and U be an open neighborhood of a point $p \in M$. Suppose that M' is compact and the local automorphism group of M' at each point $q \in M'$ is compact. Let ϕ be a biholomorphic mapping of M such that $\phi(U \cap M) \subset M'$. Then ϕ is analytically continued along any chain γ passing through the point p .*

Proof. Suppose that the assertion is not true. Then there is a chain-segment $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = p$ and ϕ can be biholomorphically continued along all subpath $\gamma[0, \tau]$ with $\tau < 1$, but not along the whole path.

Because M' is compact, there exists the limit

$$q \equiv \lim_{\tau \rightarrow 1} \phi(\gamma(\tau)) \in M'.$$

By Lemma 4.2, the subarc $\phi \circ \gamma : [0, \tau] \rightarrow M'$ is a chain-segment for all $\tau < 1$. Then, by Theorem 4.21, there exists a chain Γ' on M' such that

$$\lim_{\tau \rightarrow 1} \phi(\gamma[0, \tau]) \subset \Gamma',$$

where the condition (4.15) in Theorem 4.21 is satisfied because the local automorphism group of M' at each point $q \in M'$ is compact.

Without loss of generality, we may assume that M' is in Moser-Vitushkin normal form with the chain Γ' in the u -curve. Since $\gamma : [0, 1] \rightarrow M$ is a chain-segment, there is a chain Γ on M such that

$$\gamma[0, 1] \subset \Gamma.$$

Note that $\phi(U \cap \gamma[0, 1]) \subset \Gamma'$. Then, by Theorem 4.17, the mapping ϕ is biholomorphically continued along the chain Γ . Since the point $\gamma(1)$ is an interior point

of Γ , ϕ is biholomorphically continued on an open neighborhood of the point $\gamma(1)$. This is a contradiction. This completes the proof. \square

Theorem 5.7 (Pinchuk, Ezhov-Kruzhilin-Vitushkin). *Let M, M' be nonspherical connected analytic real hypersurfaces in complex manifolds such that M' is compact and every local automorphism group of M' at each point is compact. Suppose that there exist an open neighborhood U of a point p of M and a biholomorphic mapping ϕ on U such that $\phi(U \cap M) \subset M'$. Then the mapping ϕ is biholomorphically continued along any path in M .*

Proof. Suppose that the assertion is not true. Then there is a path $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = p$ and the mapping ϕ can be biholomorphically continued along all subpath $\gamma[0, \tau]$ with $\tau < 1$, but not along the whole path.

Let V be an open neighborhood of the point $q = \gamma(1)$. Then, by Theorem 4.8, there are a real number $\delta > 0$ and a point $x \in V \cap M$ such that $B(q; \delta) \subset V$ and, for each given curve $\eta : [0, 1] \rightarrow B(q; \delta) \cap M$, there is a continuous family of chain-segments

$$\Gamma : [0, 1] \times [0, 1] \rightarrow V \cap M$$

where $\Gamma(s, \cdot) : [0, 1] \rightarrow V \cap M$ is a chain-segment of M for each $s \in [0, 1]$ satisfying

$$\Gamma(s, 0) = q \quad \text{and} \quad \Gamma(s, 1) = \eta(s) \quad \text{for all } s \in [0, 1].$$

We take τ such that $\tau < 1$ and $\gamma[\tau, 1] \subset B(q; \delta) \cap M$. Then there is a continuous family of chain-segments

$$\Gamma : [\tau, 1] \times [0, 1] \rightarrow V \cap M$$

such that

$$\Gamma(s, 0) = x \quad \text{and} \quad \Gamma(s, 1) = \gamma(s) \quad \text{for all } s \in [\tau, 1].$$

By Lemma 5.6, the germ ϕ_{γ_τ} at the point $\gamma(\tau)$ is analytically continued to a germ ϕ_x at the point x along the chain-segment $\Gamma(\tau, \cdot)$. Then, by Lemma 5.6, the germ ϕ_x is analytically continued to a germ ϕ_{γ_s} at each point $\gamma_s \in \gamma[\tau, 1]$ along the chain-segments $\Gamma(s, \cdot)$, $s \in [\tau, 1]$.

We claim that the germs ϕ_{γ_s} , $s \in [\tau, 1]$, are the analytic continuations of the germ ϕ_{γ_τ} at the point $\gamma(\tau)$ along the subarc $\gamma[\tau, 1]$. Otherwise, there would exist a number r , $\tau < r \leq 1$, such that the germs ϕ_{γ_s} , $s \in [\tau, r)$, are analytic continuations of the germ ϕ_{γ_τ} at the point $\gamma(\tau)$, but the germ ϕ_{γ_r} is not an analytic continuation of the germ ϕ_{γ_τ} . By the way, the germ ϕ_{γ_r} is an analytic continuation of the germ ϕ_x . Note that the chain-segment $\Gamma(r, \cdot)$ is compact. Thus there is a number $\varepsilon > 0$ such that each germ $\phi_{\Gamma(r, t)}$, $0 \leq t \leq 1$, at the point $\Gamma(r, t)$ converges absolutely and uniformly on the open ball $B(\Gamma(r, t); \tau)$. Then we can find a number r_1 , $\tau < r_1 < r$, such that

$$\Gamma(r_1, [0, 1]) \subset \bigcup_{0 \leq t \leq 1} B(\Gamma(r, t); \tau).$$

Note that the germ $\phi_{\gamma_{r_1}}$ is an analytic continuation of ϕ_x along the chain-segment $\Gamma(r_1, \cdot)$ and, at the same time, it is an analytic continuation of ϕ_{γ_r} along the subarc $\gamma[r_1, r]$. Then, necessarily, ϕ_{γ_r} is an analytic continuation of ϕ at the point $\gamma(\tau)$ along the path $\gamma[\tau, r]$. This contradiction proves our claim.

Therefore, the mapping ϕ is analytically continued to an open neighborhood of the point $q = \gamma(1)$ along the path $\gamma[0, 1]$. This is a contradiction. This completes the proof. \square

Theorem 5.8 (Pinchuk). *Let D, D' be bounded strongly pseudoconvex domains in \mathbb{C}^{n+1} with simply connected real-analytic boundaries. Suppose that there is a connected neighborhood U of a point $p \in \partial D$ and a biholomorphic mapping ϕ on U such that $\phi(U \cap \partial D) \subset \partial D'$. Then ϕ extends to a biholomorphic mapping of D onto D' .*

Proof. Suppose that ∂D is a nonspherical real hypersurface. Then, by Theorem 4.11, ∂D is nonspherical as well. By Theorem 5.7, ϕ analytically extends along any path on ∂D . Since ∂D is simply connected, by the monodromy theorem, ϕ analytically extend to an open neighborhood of ∂D as a local biholomorphic mapping. Since ∂D is compact, $\phi : \partial D \rightarrow \partial D'$ is a covering map. Since $\partial D'$ is simply connected, $\phi : \partial D \rightarrow \partial D'$ is a simple covering map so that there is a biholomorphic inverse $\phi^{-1} : \partial D' \rightarrow \partial D$. Then, by Hartogs extension theorem, ϕ, ϕ^{-1} analytically extend to open neighborhoods respectively of $\overline{D}, \overline{D}'$.

Suppose that ∂D is a spherical real hypersurface. Then, by Theorem 4.11 and Lemma 4.12, $\partial D'$ is spherical as well. By Theorem 5.2, the domains D, D' are both biholomorphic to an open ball B^{n+1} so that D is biholomorphic to D' . This completes the proof. \square

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